Solutions to problem sheet 3

640-381: Sensors

September 9, 2004

1 Question 1

1.1 Problem

A copper wire of diameter 1.8 mm carries a steady current of 1.3 A. In Cu, each atom contributes one freeelectron, so the number of electrons per unit volume, n, is the same as the number of atoms per unit volume.

For comparison, consider a strip of n-type Si (doped with phosphorous) which is 3.5 mm wide and 250 μm thick, and carries a current of 5.2 mA. For Si, n=$1.5 \times 10^3 electrons/m^3$.

For both these cases,

- What is the current density?
- What is the drift speed of conduction electrons?
- What is the mean free time τ between collisions ?

Data:
Density

$$\rho_{Cu} = 9 \times 10^3 kg/m^3$$

Restivity

$$\rho_{Cu} = 1.69 \times 10^{-8} \Omega.m$$
$$\rho_{Si} = 8.7 \times 10^{-4} \Omega.m$$

1.2 Solution

We will revisit this problem when we do solid-state detectors and a little more semiconductor theory in a few weeks.

2 Question 2

2.1 Problem

Restivity, and thus resistance, vary with temperature according to

$$\rho(T) = \rho_0[1 + \alpha(T - T_0)]$$

where $\rho_0 = 1.69\mu\Sigma.cmwhenT_0 = 293K$

A secondary thermometric standard for measuring temperature in the range 14-900 K on the International Temperature Scale is a platinum resistance thermometer ($\alpha_{Pt} = 3.9 \times 10^{-3}K^{-1}$). For a Pt wire thermometer of diameter 0.1 mm and length 3 cm, what is the resistance at 14 K and 900 K?

A Pt resistive temperature detector (RTD) has a resistance given by

$$R = R_0(1 + 39.08 \times 10^{-4}T - 5.8 \times 10^{-7}T^2)$$

where $R(T = 0°) = 100\Omega$.

What is the difference between the linear and second order resistance values at $T = 150°$? To how large an error in temperature is this equivalent?

2.2 Solution

$$\begin{aligned} \rho(14K) &= 1.69[1 + 3.9 \times 10^{-3}(14 - 293)] \\ &= -0.704\Omega.cm \end{aligned}$$

Yikes! This negative restivity is unphysical, so this result tells us that the equation is not valid at such low temperatures. In fact, a new theory of restivity is needed at low temperatures. A bit of a trick question.

$$\begin{aligned} \rho(900K) &= 1.69[1 + 3.9 \times 10^{-3}(900 - 293)] \\ &= 5.69\Omega.cm \end{aligned}$$

$$
\begin{aligned}
R(900K) &= \frac{\rho L}{A} \\
&= \frac{(5.67)(3)}{\pi(0.005)^2} \\
&= 0.217\Omega
\end{aligned}
$$

The resistance expression has a linear component and a smaller quadratic component. Note that we are not told that the RTD is a wire, so we can't necessarily use the expression for the restivity of platinum wire. The second order part is

$$5.8 \times 10^{-7}(150)^2) = 1.305\Omega$$

$$R(150) = 100(1 + 39.08 \times 10^{-4}(150) - 1.305 = 159.925$$

$$R(1) = 100(1 + 39.08 \times 10^{-4}(1) - (something really tiny)) = 0.3908\Omega$$

1 degree change in temperature leads to 0.3908 Ohms change in resistance, so a 1.305 Ohm resistance change is equivalent to a 3.34 degree chaneg in temp.

3 Question 3

3.1 Problem

Consider heating a copper rod. Its resistance, length and cross-sectional area change. If the temperature changes by 1 degree C, what percentage changes in R, L and A occur? What conclusions do you draw?

Data:
The change in dimension with temperature is given by the linear coeff of thermal expansion of Cu, confusingly also called α, and with the same units!.$\alpha_{Cu} = 1.7 \times 10^{-5}K^{-1}$

3.2 Solution

We have two effects that will change the resistance of a bar of metal- one is the increase in restivity that arises from the increased number of collisions between the 'gas' of conduction electrons as the temperature increases. This is dependant only on temperature, and not on the dimensions of the wire. The second effect is the increase in the resistance of the wire because of the thermal expansion of the metal. Even if the restivity of the metal didn't change with temperature, the resistance would. The point of this problem is to calculate the two effects seperately and compare them.

let us say the rod has dimensions length c, height b, width a. The cross-sectional area is then ab. When the rod is heated, it expands in each dimension by an amount proportional to the length, ie:

$$a_{hot} = a_{cold} + \Delta a_{cold} = a_{cold} + \alpha \Delta T a_{cold} = (1 + \alpha \Delta T) a_c old$$

similarly,

$$b_{hot} = (1 + \alpha \Delta T) b_c old$$
$$c_{hot} = (1 + \alpha \Delta T) c_c old$$

and so

$$\begin{aligned} R_{hot} &= \rho \frac{c_{hot}}{a_{hot} b_{hot}} \\ &= \rho \frac{(1 + \alpha \Delta T) c_{cold}}{(1 + \alpha \Delta T) a_{cold} (1 + \alpha \Delta T) b_{cold}} \\ &= R_{cold} \frac{1}{1 + \alpha \Delta T} \end{aligned}$$

Now the percentage changes in R, L and A can be calculated, noting that $\Delta T = 1$:

$$\begin{aligned} L_{hot} &= L_{cold} + \Delta L_{cold} \\ &= L_{cold} + \alpha \Delta T L_{cold} \\ &= (1 + \alpha) L_c old \end{aligned}$$

percentage change in L is

$$\frac{\Delta L}{L} = \alpha = 1.7 \times 10^{-3} \%$$

$$\begin{aligned} A_{hot} &= A_{cold} + \Delta A_{cold} \\ &= (a + \Delta a)(b + \Delta b) \\ &= a(1+\alpha)b(1+\alpha) \\ &= A(1+\alpha)^2 \end{aligned}$$

$$\frac{\Delta A}{A} = ((1+\alpha)^2 - 1) \sim 2\alpha$$

Note that this can be simplified by neglecting the α^2 term in the expansion. By a similar process,

$$\frac{\Delta R}{R} = (1 - (1+\alpha)^2) \sim -2\alpha$$

If we compare $1.7 \times 10^{-3}\%$ with the change in resistance due to restivity from above (0.3908/160=0.24%) we see that the change in resistance from the dimensional change alone is negligible when compared with the change due to the change in restivity.

4 Question 4

4.1 Problem

A thin Film strain Gague (of film thickness 250 μm, containing a 78 mm long Constantan wire) is bonded to a 1x1 m square concreete support for a highway bridge. If the resistance of the sensor increases by 20%, what weight is bearing down on the column?

4.2 Solution

The most sane way to tackle this problem is to make a few simplifying assumptions. Let us say that the concrete will squish up by an amount dictated by its young's Modulous, and that this will in turn squish the wire, reducing its cross-sectional area. Let us simplify the calculation by assuming that the length of the wire will not change, and that all the change in resistance is due to the wire becoming thinner. Let us assume even more, and say that the resistivity will not change as a result of the compression.

Under these assumptions, all we really need is the Young's modulus and the percentage reduction in resistance.

$$
\begin{aligned}
R &= \tfrac{4\rho L}{\pi x^2}) \\
R &= (constantswemayaswellignorebecausetheywillcancelinafewsteps)\frac{1}{x^2} \\
\Delta R &= \frac{1}{(x+\Delta x)^2} - \frac{1}{x^2} \\
&= (rearrange, expandandthrowawaysecondorderterms) \\
\frac{\Delta R}{R} &= \frac{-2\Delta x}{x+2\Delta x} \\
&= 0.2(weknowresistancedropsby20percent)
\end{aligned}
$$

This gives us

$$\frac{\Delta x}{x} = 0.083 = \frac{mg}{Y}$$

and we find mg easily.

5 Question 5

5.1 Problem

A 1 mm thick piezoelectric sensor with an area of 1x1 cm develops a voltage of 1.6 kV. If the sensor is made of PVDF, what weight is being applied to the sensor? What voltage would it generate if the sensor was made of $BaTiO_3$?

Data:

$$
\begin{aligned}
\epsilon_r^{BaTiO_3} &= 1700 \\
\epsilon_r^{PVDF} &= 12 \\
d_{11}^{BaTiO_3} &= 78pC/N \\
d_{11}^{PVDF} &= -30pC/N
\end{aligned}
$$

5.2 Solution

The piezoelectric effect generates a charge on some of the faces of a crystal, usually a pair of opposite sides. Sometimes the charged surfaces are the ones to which the force is being applied; other times it is a differnt pair of faces. The subscripts n and m on the d_{nm} tell us which faces develop the charge, but in these simple problems we don't care. The charge generated is

$$Q = d_{11}F$$

we relate the charge generated to the voltage generated by the good 'ol capacitor equation for a slab:

$$Q = CV$$

and

$$C = \frac{\epsilon A}{d}$$

combine to get (noting that d is the thickness and d_{11} is the piezo constant):

$$\begin{aligned} V &= Q/C \\ &= \frac{d_{11}Fd}{\epsilon A} \end{aligned}$$

Plug in V and solve for F to get

$$F = 0.563N$$

plug back in and use the constants for $BaTiO_3$ to get

$$V = 29.3V$$

6 Question 6

6.1 Problem

A 1 mm thick $BaTiO_3$ pyroelectric sensor is subject to a 50 K temperature change. What voltage does is generate? If the same voltage was generated from a PVDF sensor of the same dimensions, what temperature change would that represent?

Data:

$$\epsilon_r^{BaTiO_3} = 1700$$
$$\epsilon_r^{PVDF} = 12$$
$$P_Q^{BaTiO_3} = 4 \times 10^{-4} C.m^{-2} K^{-1}$$
$$P_Q^{PVDF} = 4 \times 10^{-3} C.m^{-2} K^{-1}$$

6.2 Solution

There is quite a bit of ambiguity in the literature about the proper formula to use-but the best way forward is to use dimensional analysis and a bit of nous. P_Q has units of $C.m^{-2}K^{-1}$, so it tells us how many coulombs of charge are generated on every square metre of the surface of the pyroelectric slab perdegree rise in temperature. Hence,

$$Q = P_Q A \Delta T$$

we relate the charge generated to the voltage generated by the good 'ol capacitor equation for a slab:

$$Q = CV$$

and

$$C = \frac{\epsilon A}{d}$$

combine to get

$$\begin{aligned} V &= Q/c \\ &= \frac{d P_Q A \Delta T}{\epsilon A} \\ &= \frac{d P_Q \Delta T}{\epsilon} \\ &= \frac{(.001)(.0004)(50)}{(8.8 \times 10^{-12})(1700)} \\ &= 1336 Volts \end{aligned}$$

which is a heck of a lot! Going backwards for the second part of the problem:

$$\begin{aligned}
1336 &= \frac{dP_Q \Delta T}{\epsilon} \\
&= (.001)(.0004)(\Delta T)(8.8 \times 10^{-12})(12) \\
\Delta T &= 3.53K
\end{aligned}$$

7 Question 7

7.1 Problem

7.2 Solution

$$\begin{aligned}
&= \\
&= \\
&=
\end{aligned}$$