An integral involving four Hermite polynomials

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We wish to find a computationally efficient solution to

$$I^{1/2}_{abcd} = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2/(1/2)} \mathrm{H}_a(x)\mathrm{H}_b(x)\mathrm{H}_c(x)\mathrm{H}_d(x) \tag{1}$$

where $\mathrm{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ are the (physicists') Hermite polynomials.

By symmetry of the Hermite polynomials, for $a+b+c+d$ odd, $I^{1/2}_{abcd} = 0$. This is assumed in the following discussion.

[Witschel(1973)] gives a derivation for any number of Hermite polynomials using commutation algebra. Unfortunately the procedure cannot be performed for general a, b, c, d. This article follows the procedure of [Titchmarsh(1948)] and essentialy performs one more iteration, extending the result for three Hermite polynomials to four. An alternative method is given in [Busbridge(1948)], which gives a general result for $I^x_{a_1 a_2 \cdots a_n}$.

To do this, we wish to get (1) into the form of Titchmarsh's result

$$\begin{aligned} I^{1/2}_{abc} &= \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-2x^2} \mathrm{H}_a(x)\mathrm{H}_b(x)\mathrm{H}_c(x) \\ &= \frac{2^{1/2(a+b+c-1)}}{\pi} \Gamma\left(\frac{a+b-c+1}{2}\right) \Gamma\left(\frac{a-b+c+1}{2}\right) \Gamma\left(\frac{-a+b+c+1}{2}\right) \end{aligned} \tag{2}$$

Using the identity

$$\mathrm{H}_m(x)\mathrm{H}_n(x) = 2^n n! \sum_{r=0}^{n} \binom{m}{n-r} \frac{\mathrm{H}_{m-n+2r}(x)}{2^r r!} \qquad , n \leq m$$

where

$$\binom{m}{n-r} = \frac{m!}{(n-r)!(m-n+r)!}\,,$$

(1) becomes

$$I^{1/2}_{abcd} = 2^d d! \sum_{r=0}^{d} \frac{c!}{(d-r)!(c-d+r)!} \frac{1}{2^r r!} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2/(1/2)} \mathrm{H}_a(x)\mathrm{H}_b(x)\mathrm{H}_{c-d+2r}(x)$$

and then using Titchmarsh's result (2) gives

$$\begin{aligned} I^{1/2}_{abcd} = 2^d d! \sum_{r=0}^{d} &\frac{c!}{(d-r)!(c-d+r)!} \frac{1}{2^r r!} \frac{2^{1/2(a+b+c+d-1)}}{\pi} \times \\ &\Gamma\left(\frac{a+b-c+d+1}{2} - r\right) \Gamma\left(\frac{a-b+c-d+1}{2} + r\right) \Gamma\left(\frac{-a+b+c-d+1}{2} + r\right) \end{aligned} \tag{3}$$

which, after factoring out Γ-functions with the r omitted, can be written as

$$\begin{aligned} I^{1/2}_{abcd} = &\frac{2^{(a+b+c+d-1)/2}}{\pi} \Gamma\left(\frac{a+b-c+d+1}{2}\right) \Gamma\left(\frac{a-b+c-d+1}{2}\right) \Gamma\left(\frac{-a+b+c-d+1}{2}\right) \times \\ &\frac{c!}{(c-d)!}\, {}_3F_2\left(-d, \frac{(c-d)+a-b+1}{2}, \frac{(c-d)-a+b+1}{2}; 1+c-d, \frac{(c-d)-a-b+1}{2}; 1\right) \end{aligned} \tag{4}$$

Unfortunately, unlike the fewer-Hermite-polynomial integrals, there is no known identity which allows the hypergeometric function ${}_3F_2\left(\ldots\right)$ to be expressed in terms of Γ-functions. The author believes it is quite likely that there is such an identity. For example, Saalschütz's Theorem,

$$ {}_3F_2\left(-x,-y,-z;\,n+1,-x-y-z;\,1\right)=\frac{\Gamma\left(n+1\right)\Gamma\left(x+y+n+1\right)}{\Gamma\left(x+n+1\right)\Gamma\left(y+n+1\right)} $$

comes extremely close. It is also observed that such an identity would likely make (4) symmetric in a, b, c and d. (3) is a computationally fast solution as Γ-functions can be calculated efficiently with the Lanczos approximation or even exactly for half-integer arguments

$$ \Gamma\left(n+\frac{1}{2}\right)=\sqrt{\pi}\frac{(2n-1)!!}{2^n}=\sqrt{\pi}\frac{(2n)!}{2^{2n}n!} $$

References

[Witschel(1973)] W. Witschel. Harmonic oscillator integrals. *Journal of Physics B: Atomic and Molecular Physics*, 6:527–534, 1973.

[Titchmarsh(1948)] E. C. Titchmarsh. Some integrals involving hermite polynomials. *J. London Math. Soc.*, 23:15–16, 1948.

[Busbridge(1948)] I. W. Busbridge. Some integrals involving hermite polynomials. *J. London Math. Soc.*, 23:135–141, 1948.