Lecture 4	Krane	Enge	Cohen	Williams
the Deuteron	Ch. 4	Ch. 2	Ch 3	9.8
d-wave admixture	4.1	2.6	3.5	9.8
tensor force	4.1	2.6	3.5	9.8
missing S^1 state	4.4	2.5	3.4	
isospin	11.3	6.7	3.4	9.7

Problems on Lecture 4

1 What is the minimum photon energy required to dissociate the deuteron? Take the binding energy to be 2.224589 MeV.

2 If the binding energy of the deuteron was 10 MeV instead of 2.23 MeV, what would be the approximate depth of the assumed square-well potential? Estimate the value of r_d.

3 Assume the deuteron wave function in the region of the well, is given by

$$u(r)_{ii} = A\sin K(r-c) \quad where \;\; K = \frac{1}{\hbar}\sqrt{2m(E_b - V_0)}$$

and by $u(r)_{iii} = Be^{-K'r} \; where \, K' = \frac{1}{\hbar}\sqrt{2mE_b}$ in region iii.

What fraction of the time do the proton and neutron spend outside the range of their forces?

Lecture 3 Review

1 Semi-empirical mass formula

Understand the origin of the terms
Do a calculation to find the most stable isobar

2 Other properties

a Angular momentum and parity I^{π} of GS and excited states
For simple nuclei ***I*** is result of ***j*** of component nucleons
For odd A ***I*** is half integral (one unpaired nucleon)
For even A ***I*** is integral

b Magnetic dipole moment of nucleus

$\mu = g\ell\mu_N$ where $\mu_N = \frac{e\hbar}{2m}$ m is nucleon mass and ℓ is the AM QNo.
NDM due to spin $\mu = g_s s \mu_N$
significance of g_s for proton and neutron regarding substructure of nucleon

c Nuclear shape *(actually covered in lecture 4)*

Quadropole moment (measure of deformation of nucleus)
$Q = 1/e\int \rho(3z^2 - r^2)dV]$ and has units of area usually in barn (10^{-28} m^2).
Q +ve prolate (football shape)
Q –ve oblate (cushion)

Lecture 4

I think we have spent enough time discussing the general properties of nuclei to give us the basic information we need to discuss the details of the force between nucleons.
Firstly let's summarise what we know about the nuclear force:

Some properties of the Nuclear Force

observation	implication
Constant nucleon density	**Short-range force with a repulsive core.**
Few odd-odd nuclei	**pairing ➔ stability**
S_n for Ca isotopes	**pairing energy ~3 MeV**

The actual properties of the nuclear potential are quite complicated. We will find it depends not only of **r,** but on the spin s of the nucleons, and their relative AM ℓ.

We can find out about the potential in two main ways. One is scattering of nucleons of each other or from nuclei. (This will be discussed a little later).

The other is to study any stable states that might result from this interaction. There are of course 3 possibilities: pn, pp, nn. It turns out that only one of these forms a stable nucleus. We will discuss why, and what this means.
In this instance we will study the nucleus ^{2}H, the deuteron, which is the nucleus formed when a proton and neutron combine.

It is the simplest nucleus and in a sense is the nuclear equivalent of the H atom in atomic studies. Unfortunately, unlike the H atom, it does not have any excited states, so there is no information available from its spectroscopy. None the less the fact that it is bound at all reveals a lot about the nuclear force. (of course if it were not bound, we wouldn't be here to study nuclear physics, since this nucleus is the basis for fusion to all the heavier nuclei in the universe)

Lets summarise what is known about the deuteron:

Binding Energy 2.225 MeV Check this from tables of masses
How do I know?
Thermal neutron capture
^{2}H(γ,n)p threshold

No evidence of excited states

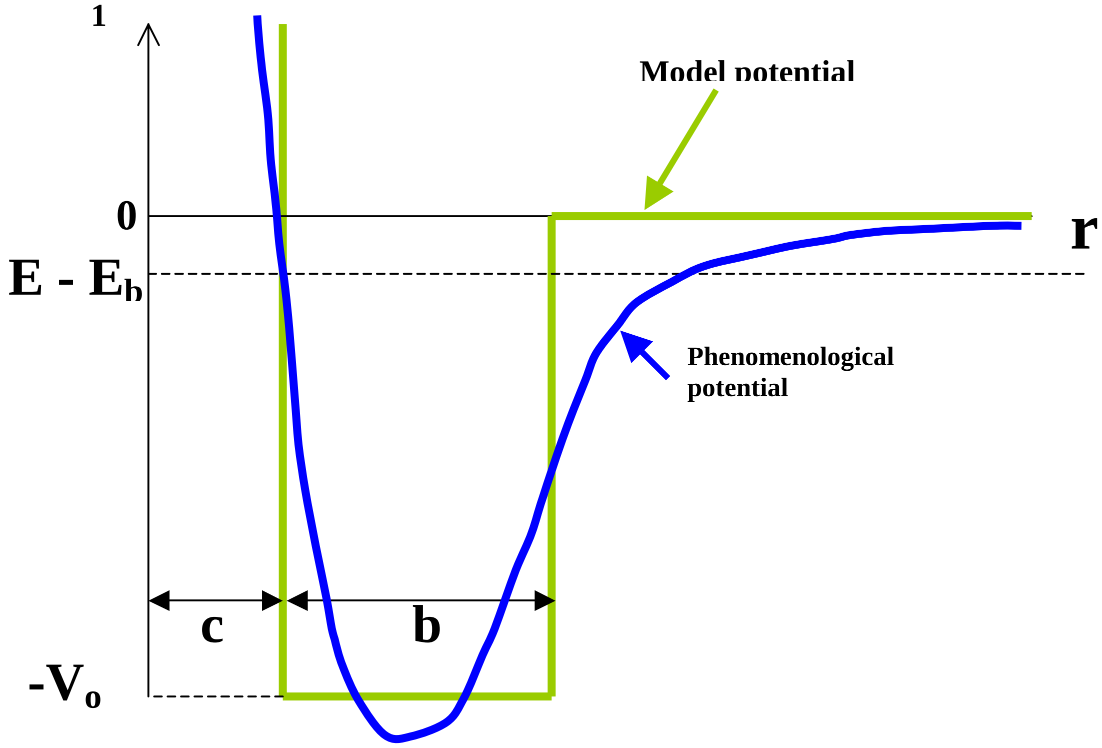
Spin I=1

Magnetic dipole moment μ_d = +0.85735 n m
(μ_p = +2.79275 n m)
(μ_n = - 1.91350 n m)

Quadrupole moment Q = 0.00282 barn (expect zero)

Radius R_d = 2.1 fm (1 fm = 10^{-15} m)

Solution
We need to solve the Schrodinger wave equation for a potential with the characteristics we mentioned above. An attractive short-range force between the p and n, and a repulsive core.



Now the form of V may be V(rθφ), but we will assume that it is only a function of r. We should also consider the possibility that the AM of the n rel to p is not necessarily zero, in which case we would need to include a centrifugal component in V which looks like

$V = \dfrac{l(l+1)\hbar^2}{2mr^2}$. In fact since we know that there are no excited states we will assume that l=0

Note that this is the potential that the proton feels due to the n, or vice versa. The solution of this 2-body problem can be resolved down to the solution of a single particle moving in this potential, using the radial scale as the separation of one from the other, if we work in the CM system and replace the mass of the particle with the reduced mass of the two.

$$m = \frac{m_p m_n}{m_p + m_n} = \frac{1}{2} m_n$$

Thus we are to solve

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r\theta\varphi) = (E - V)\psi(r\theta\phi)$$

where, $\psi(r\theta\phi) = u_\ell / r Y_{\ell m}(\theta\phi)$

However if we assume no angular dependence $Y_{\ell m} \rightarrow (1/4\pi)^{1/2}$, and we can solve the 1-D WE for u(r).

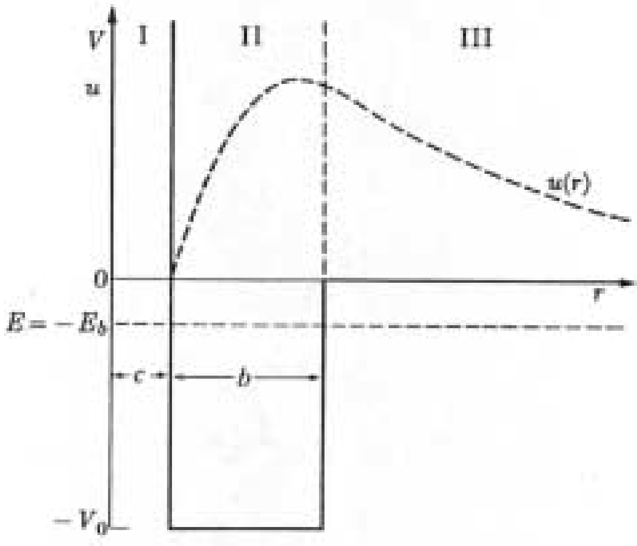
for r<c u(r) = 0 since V = ∞

for c< r <c+b

$$u(r)_{ii} = A\sin K(r-c) \quad where \;\; K = \frac{1}{\hbar}\sqrt{2m(E_b - V_0)}$$

for r> c+b

$$\boldsymbol{u(r)_{iii} = Be^{-K'r} \;\; where \;\; K' = \frac{1}{\hbar}\sqrt{2mE_b}}$$



The complete continuous solution requires continuity of both $u(r)$ *and* $\frac{du(r)}{dr}$ at r = c+b

$$A\sin Kb = Be^{K'(c+b)}$$

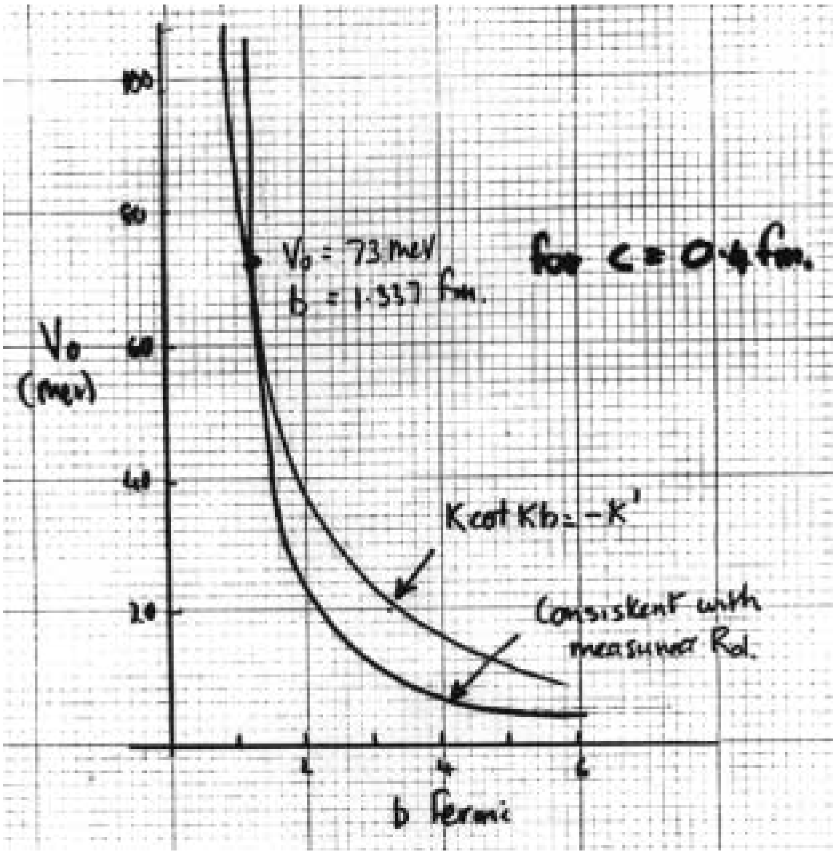
$$and$$

$$AK\cos Kb = -K'Be^{-K'(c+b)}$$

$$dividing \;\; these \;\; gives$$

$$K\cot Kb = -K'$$

K depends on V and E_b and K‘ depends on E_b.



This transcendental equation gives a relationship between b and V that is consistent with E_b. That is we can a range of values of V and b that will give the correct BE for the deuteron We need more information to define either the depth V or the width b.

In fact the radius of the deuteron has been measured, and on the vugraph I have written out the calculation of rms radius of the deuteron from our solution.

We can calculate the deuteron radius from our solution.

1st need to find A and B by normalizing $\int \varphi^2 d\tau = 1$ **(see Enge page 41)**

This gives $A^2 = 1 + \frac{2K'}{1+K'b}$

$$B^2 = \frac{2K'(\sin^2 Kb)e^{2K'(c+b)}}{1+K'b}$$

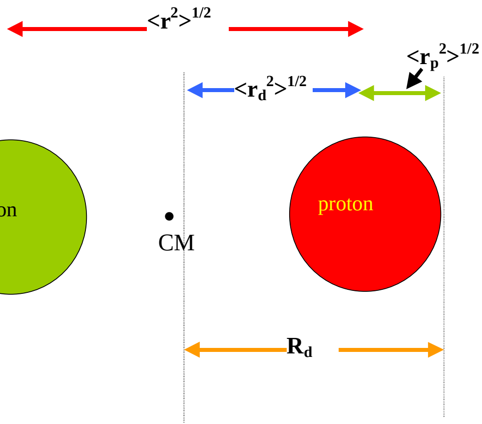
$$<r_d^2> = \int_0^\infty r^2 \varphi^2 d\tau \quad \textit{where } \varphi = \frac{u(r)}{r} Y_{\ell m} = \frac{u(r)}{r}\sqrt{\frac{1}{4\pi}}$$

$$= \int_0^\infty r^2 u^2 \, dr \qquad \text{since } d\tau = 4\pi r^2 dr$$

$$<r_d^2> = \frac{1}{8K'^2} - \frac{1}{8K^2} + \frac{(2c+b)(1+K'b)}{K'} + \frac{c^2}{4} - \frac{1}{24(}$$

Known, so this provides another relation between V_o and b

C = 0.4 fm
V_o = 73 MeV
B = 1.337 fm



R_d is the measured radius of deuteron = 2.1 fm

$R_d^2 = <r_d^2> + <r_p^2>$ $\qquad$ $<r_p^2>^{1/2} = 0.8$ fm

Where $<r_d^2>^{1/2} = ½<r^2>^{1/2}$

This gives another relation between E_b and b, which leads to a well depth of 73 MeV.

Note that in order to solve the equation we need to set a value for the width of the repulsive core. The value chosen is c = 0.4 fm. So that the deuteron is bound with the correct BE if

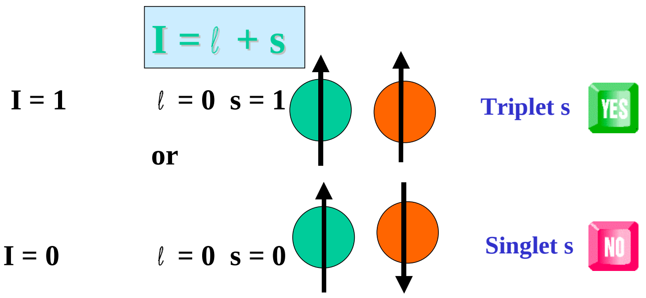
c = 0.4 fm
V_o = 73 MeV and the width of the well is
b = 1.337 fm

However you should note that the nuclear potential is not a square well, and that this calculation is only a first approximation to the real situation.
A very important characteristic of the nuclear force is evident from the deuteron. Firstly there is only one bound state of the deuteron. It has I=1. We assumed that *l*=0 so it must be that the n and p have their spins aligned, giving S=1. This is known as the triplet S state (there are 3 projections of S=1).

But if the nuclear force depended only on the spatial co-ordinate r, it should make no difference if the spins were parallel or anti-parallel. Thus we would expect at the same energy an S=0 state (ie the GS should be degenerate). There is no S=0 state (singlet S)

We assumed that the n and p had a relative AM ℓ =0

The AM of the deuteron is



This must mean that the nuclear potential is weaker for two nucleons aligned anti-parallel that aligned parallel. Or in other words the potential well is less deep for anti-parallel nucleons. How much shallower?

Conclusion:

The nuclear force is stronger for parallel spins

On the basis of this model, it can be shown that the minimum depth of the well for the given value of b, that will just bind the p and n is about 60 MeV.

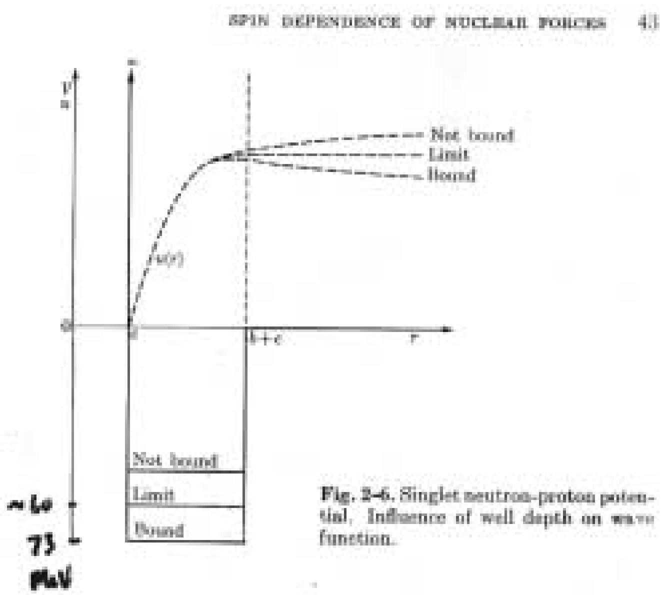


Fig. 2-6. Singlet neutron-proton potential. Influence of well depth on wave function.

So it seems that there is a significant difference in the nuclear potential. From scattering studies, which we will look at shortly we know that the singlet state is only just unbound (by about 60 keV). We should remember this information when we discuss the various terms that contribute to the nuclear potential.

There is another worry about our assumption (calculation) that the deuteron wavefunction is l=0. It comes from examination of the magnetic dipole moment of the deuteron. Because the l =0 wave function is spherically symmetric, the proton cannot contribute to the mag. moment by virtue of its orbital motion (l =0 in semi classical terms means a degenerate ellipse). So the MM of the deuteron should be simply the sum of the intrinsic MM of the proton and neutron (this is due to their spin).

Magnetic dipole moment of deuteron

	mag. mom. (nuclear magnetons)
Proton	**2.79275**
neutron	**-1.91350**
deuteron expected	**0.87925**
deuteron observed	**0.85735**
discrepancy	**0.02190 (~2.5%)**

If ℓ = 0 the proton cannot produce a magnetic field

The ℓ = 0 wave function is spherically symmetric

(or the ℓ = 0 semi-classical orbit is a straight line)

Conclusion:

The wave function must have a component with ℓ >0

This discrepancy is significantly greater than the errors in measurement. There may be several reasons for this discrepancy, including the possibility that the proton may be affected by the close proximity of the neutron within the nucleus of deuterium. In fact of binding the p and n together invokes, as we shall see, the exchange of π mesons. These indeed may have charge (π^+, π^-) and contribute to the MM.

However the likely answer is that the WF is not purely l=0., there is probably some ADMIXTURE of l = 2. (a proton in an l =2 orbit of course can be seen as a current sweeping out an area, and producing a MM). In which case we should see the WF of the deuteron as:

$$\psi = a_s\psi_s + a_d\psi_d$$

4% ℓ = 2 contribution

$$\Psi = a_s\Psi_s + a_d\Psi_d$$

ℓ = 0 ℓ = 2

Why no ℓ = 1?

parity is $\pi = (-1)^{\ell}$

All the terms in ψ must have same parity

ψ_1 has ℓ =1 and hence negative π,

ψ_s and ψ_d have positive π.

But why have we chosen the admixture as l=2; a d-state impurity? What about l=1; a p-state impurity? Well, the parity of the GS WF is even, and the parity of an l=1 WF is neg (Parity = $(-1)^{\pi}$.) So in order to maintain the parity of the WF the impurity has to be of even l.

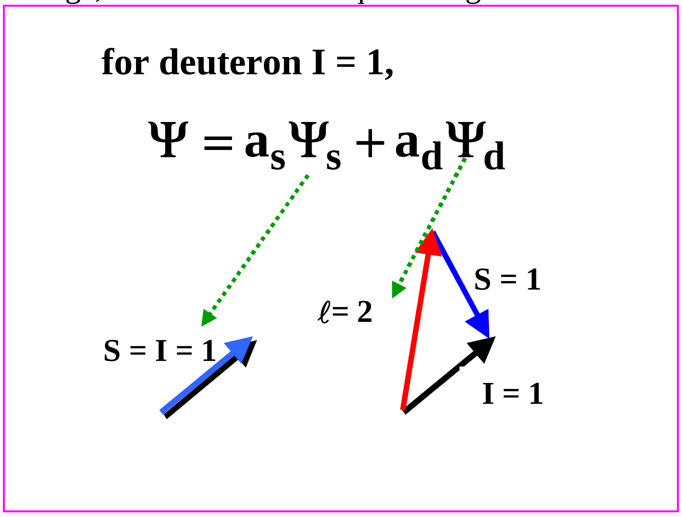
This means that the deuteron spends most if the time in an S state and a small fraction of the time in a D state. This fraction specifies $(a_d)^2$. When it changes from S to D the spin of one nucleon has to flip over in order to conserve AM.

The exact size of a_d that gives the observed MM is about 4%.

However there is clear evidence that our assumption that the deuteron is a pure S state is incorrect. This comes from the measured electric quadrupole moment.

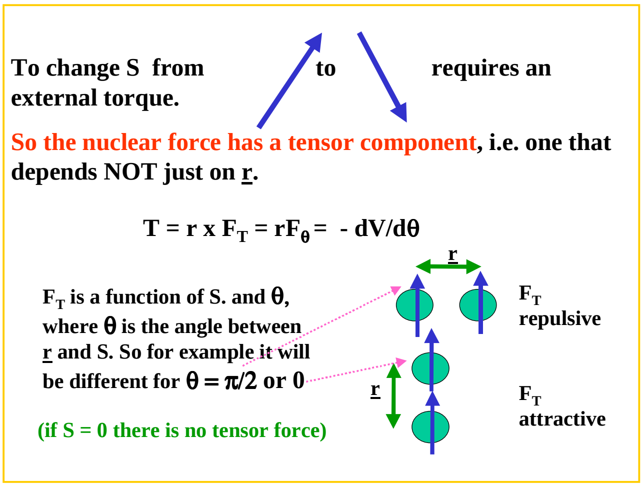
For an S state, Q=0, since the wave function is spherically symmetric. Yet the measured Quadrupole moment is Q = 0.00288 barn. So there must be a WF component that is not spherically symmetric.

Well if we allow the orbital (but not the total since the spins will flip over) AM to blithely change, there must be a torque acting.



$T = r \times F = r F_\theta = -dV/d\theta$.

This implies that the potential involved must be a fn of θ,. But the central potential is a fn of r only. This component of the nuclear potential is a non-central or tensor force.



Since the only direction in space that is relevant to the deuteron is the direction of the spin S, the θ on which this force depends must be measured from this direction. In vector notation the force must be a function of S.r. It will be different for $\theta = \pi/2$ or 0.

⇈ (repulsive)

and ⇈

(attractive) ⇈

Consider the analogue of two bar magnets parallel or in line. They have different energies and it requires a tensor force to move them.

When the p and n are in an S=0 state there is no tensor force

Why is there no singlet state of the deuteron?

However it is worthwhile asking why there is no S=1 (triplet) bound state of the n-n or p-p. Does this imply that the nuclear force is different between n-n (the di-neutron) or p-p (the di-proton)? We might well imagine that since the deuteron is only just bound, that the coulomb repulsion of 2 protons separated by a fermi or so might destabilise the system and be unbound: but not 2 neutral neutrons.

The short answer is that the Pauli Principle forbids it….no two identical particles can have the same quantum state (or the same set of quantum numbers). The long answer (which is really the explanation of the Pauli principle) is that the **total** wave function for identical particles must **be anti-symmetric.**

The wave function that we obtained for the p-n system looked like:

$$\psi(space) = \frac{u_l(r)}{r} P_{lm}(\theta) e^{im_l\phi}$$

We made the assumption that *l* =0 (that is the wave function is s-state) so the WF became spherically symmetric and the angular part became $(1/4\pi)^{-1/2}$. The symmetry of the WF is whether it changes from + to - on reflection through the origin. Mathematically this symmetry is related to the parity which is $(-1)^{\ell}$.

So this wavefunction is NOT anti-symmetric. This is because this is not the total WF. The spin WF is part of it so that the more complete WF is

$$\psi = \frac{u_l(r)}{r} P_{lm}(\theta) e^{im_l\phi} \chi(spin)$$

χ(spin) is symmetric if the spins are parallel (S=1 or triplet state), and antisymmetric if they are antiparallel (S=0 or singlet state).

Now let's look again at the WF we derived for the deuteron

it has ℓ=0 symmetric
S=1 symmetric
so it is NOT a valid WF.

Well there is clearly something wrong here, since this is what a deuteron looks like: one p and one n in ℓ=0 S=1.

If we were dealing with electrons (which are also fermions and conform to the same rules as nucleons) this would indeed be an invalid WF. We would have to flip one of the electrons over to make the WF antisymmetric. For electrons this would be the total WF. However for nucleons we have not yet specified the WF completely.

The total WF is:

$$\psi = \frac{u_l(r)}{r} P_{lm}(\theta) e^{im_l\phi} \chi(spin)\, \tau\ (T)$$

where τ(T) is the **isospin** part of the WF. This part of the WF is symmetric if the two nucleons are the same and antisymmetric if they are different (i.e. one p and one n).

Now we can see that our WF for the GS of the deuteron is indeed valid. We can also see that under these rules the WF for a di-neutron (nn) or di-proton (pp) is NOT valid.

However we must take this a little bit deeper and formalise this **concept of isospin.**

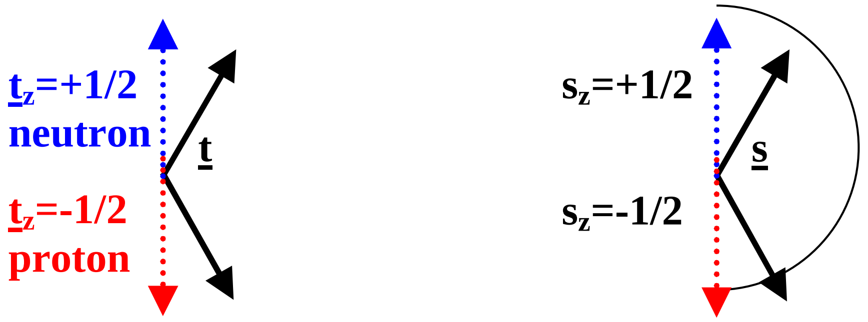
In essence we need to think of the proton and neutron as different states of the same particle; the nucleon. To specify the different states we could use coloured pens: red for protons and green for neutrons, and then these colours become the QN of the p and n. In QM we assign a new quantum number (the isospin QN) to the nucleon.

t= ½ .

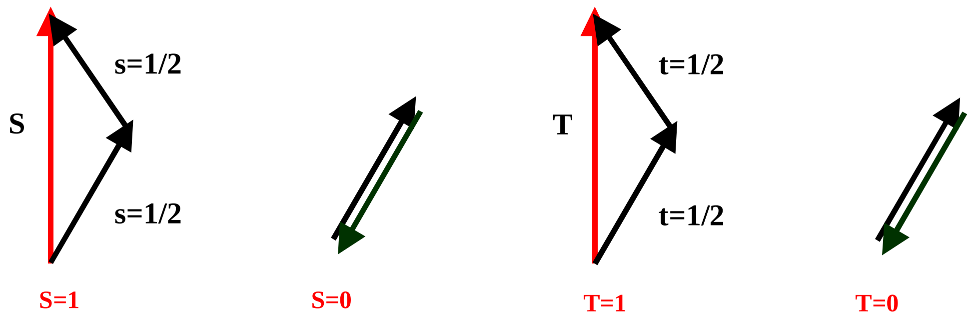
This is in exact analogy to the intrinsic spin s= ½, and just as the intrinsic spin can have projections $s_z = \pm½$, so can *t*.

t_z = + ½ corresponds to a neutron
t_z = - ½ corresponds to a proton

(note this convention is the <u>opposite</u> to that used in particle physics)

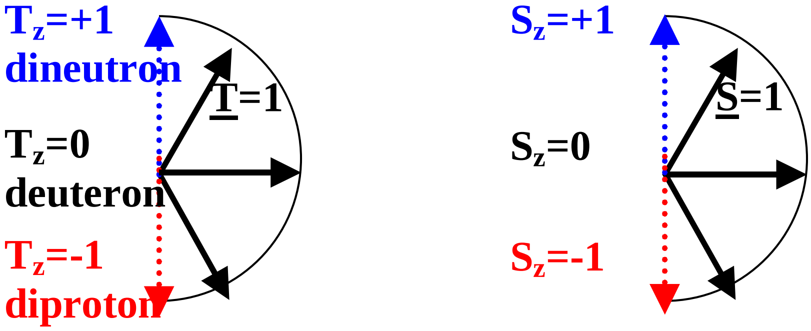


The isospin quantum state of a nucleon pair is the **vector** sum of their **isospin** . That is, the state can have T=0 or T=1, ie a singlet or triplet isospin state.



Just as the singlet S state (S=0) can have only 1 projection (S_z=0) and the triplet S state (S=1) have 3 projections (S_z = -1,0,+1);
so there is only one state associated with the T=0 (singlet) isospin state T_z=0. The T=1 (triplet) state has 3 projections T_z= -1,0,+1.

What does T_z mean? $T_z = \sum t_z$. So T_z= 0 means a p and a n.
T_z= +1 means 2 neutrons
T_z= -1 means 2 protons



Now let's look again at the 2-nucleon wavefunctions possible for the deuteron.

$\psi = \frac{u_l(r)}{r} P_{lm}(\theta) e^{i m_l \phi}$		$\chi(spin)$	$\tau\,(T)$	
1	ℓ=0	S=1	T=0	deuteron
	symmetric	symmetric	anti-sym	<u>ANTI</u>
2	ℓ=0	S=0	T=1 (T_z=0)	deuteron
	symmetric	anti-sym	symmetric	<u>ANTI</u>
3	ℓ=0	S=0	T=1 (T_z=+1)	dineutron
	symmetric	anti-sym	symmetric	<u>ANTI</u>
4	ℓ=0	S=0	T=1 (T_z= -1)	diproton
	symmetric	anti-sym	symmetric	<u>ANTI</u>

So we see that there are 4 possible combinations that are allowed for a 2-nucleon system.

1. is the bound state of the deuteron

2. is a legal state of the deuteron, but we found that this is unbound by about 60 keV. This led us to the important discovery that the nuclear central force depends on the spin orientation; the potential is shallower for anti-parallel spins.

3 and 4 are legal states, but clearly if 2 is unbound so are they.