

Lecture 5	Krane	Enge	Cohen	Williams
N-N scattering $\ell=0$	4.2/3	3.1-3	3.10	2.11, 9.9
phase shift	4.2	3.5	3.10	
Properties of 2-nucleon potential				
Central	4.4		3.6	
spin dependence	4.4	3.6	3.7	
ℓ -s dependence	4.4		3.8	
tensor component	4.4	2.6	3.6	

Problems for Lecture 5

- 1 Calculate the classical impact parameter necessary to give an orbital AM of $\ell=1$ for an n - p scattering event when $E_{lab} = 10$ MeV
- 2 Matching $\ell=0$ wavefunctions at the boundary $r = c + b$ of a square-well repulsive core potential leads to the transcendental equation $K \cot Kb = k \cot [k(c+b) + \delta]$. Determine the radius of the repulsive core (c) from the information that the triplet S phase shift is zero for $E_{lab} = 350$ MeV. Use $V_0 = 73$ MeV, and $b = 1.337$ Fm

Review Lecture 4

The Deuteron

- 1 Nuclear potential short range attractive
 Repulsive core
 Approximate with square well plus infinite repulsive core
 assume $\ell=0$
- 2 Solve Schrodinger Equation for $\ell=0$ 1D, knowing BE of deuteron as 2.23 MeV
 Transcendental equation gives binding for range of pot. depths and widths
 Calculate radius of deuteron from WF and equate with observation
 Gives another transcendental equation
 Combined, these give V_0 , width of pot. well
- 3 I for deuteron is 1 ($S=1$), but p and n can couple to give $S = 1$ (triplet S state) or 0.
 No $S=0$ (singlet) GS state found. $S=0$ state is just unbound. Therefore Pot. well is
 shallower (60 MeV) for antiparallel coupling of s.
- 4 Mag. Dipole moment less than sum of MDM for p and n (since $\ell=0$ there is no
 orbital contribution from motion of p)
 Therefore postulate that the WF for deuteron is not entirely $\ell=0$ (s-wave)
$$\psi = a_s \psi_s + a_d \psi_d^{(*)} \text{ where } \ell=2 \text{ (d wave) contribution is about 4\%}$$

 (*) Understand what this equation means
- 5 I for deuteron is 1 and since $\ell=0$, $S = s_n + s_p = \frac{1}{2} + \frac{1}{2} = 1$
 For $\ell=2$ component must recouple $S=1$ and $\ell=2$ to give $I = 1$
 This means twisting S, **so nuclear pot. must have a tensor part.** Depends on θ and S as
 well as r.
- 6 Measured Quadrupole moment is non zero, confirms deuteron must include $\ell=2$
 component in WF. (unlike $\ell=0$ WF, $\ell=2$ WF is not spherically symmetric)
- 7 Why there is no $S=0$ deuteron, and why there is not dineutron or diproton.
 Paul exclusion principle does not permit 2 identical particles to have same set of Q
 numbers.
- 8 Introduction of Isospin as a quantum number. Nucleons have $t = \frac{1}{2}$. $T_z = +1/2$ is
 neutron, $t_z = -1/2$ is a proton.

Left over from Lecture 4

The total WF is:

$$\psi = \frac{u_l(r)}{r} P_{lm}(\theta) e^{im_l \phi} \chi(\text{spin}) \tau(T)$$

where $\tau(T)$ is the **isospin** part of the WF. This part of the WF is symmetric if the two nucleons are the same and antisymmetric if they are different (i.e. one p and one n).

Now we can see that our WF for the GS of the deuteron is indeed valid. We can also see that under these rules the WF for a di-neutron (nn) or di-proton (pp) is NOT valid.

However we must take this a little bit deeper and formalise this **concept of isospin**.

In essence we need to think of the proton and neutron as different states of the same particle; the nucleon. To specify the different states we could use coloured pens: red for protons and green for neutrons, and then these colours become the QN of the p and n. In QM we assign a new quantum number (the isospin QN) to the nucleon.

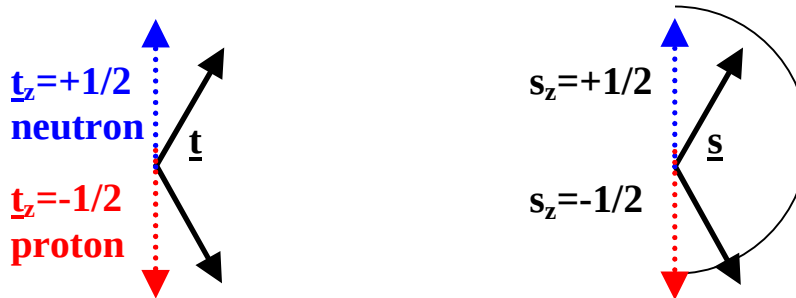
$$t = \frac{1}{2}.$$

This is in exact analogy to the intrinsic spin $s = \frac{1}{2}$, and just as the intrinsic spin can have projections $s_z = \pm \frac{1}{2}$, so can t .

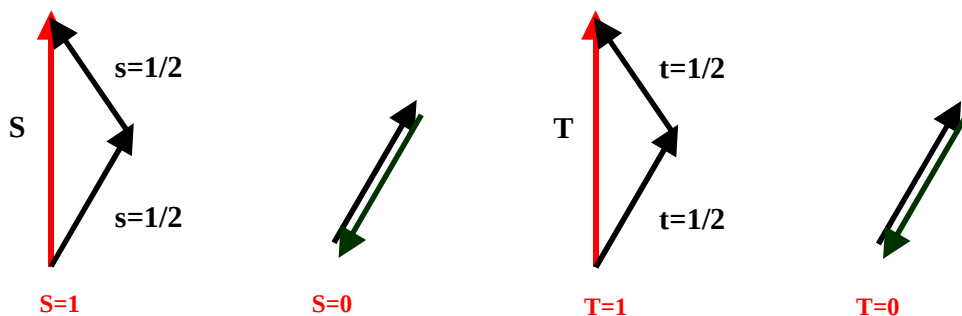
$t_z = +\frac{1}{2}$ corresponds to a neutron

$t_z = -\frac{1}{2}$ corresponds to a proton

(note this convention is the opposite to that used in particle physics)

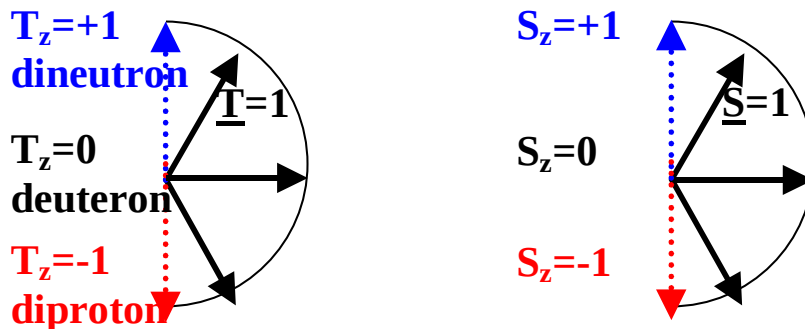


The isospin quantum state of a nucleon pair is the **vector** sum of their **isospin**. That is, the state can have $T=0$ or $T=1$, ie a singlet or triplet isospin state.



Just as the singlet S state ($S=0$) can have only 1 projection ($S_z=0$) and the triplet S state ($S=1$) have 3 projections ($S_z = -1,0,+1$);
so there is only one state associated with the $T=0$ (singlet) isospin state $T_z=0$. The $T=1$ (triplet) state has 3 projections $T_z = -1,0,+1$.

What does T_z mean? $T_z = \sum t_z$. So $T_z = 0$ means a p and a n.
 $T_z = +1$ means 2 neutrons
 $T_z = -1$ means 2 protons



Now let's look again at the 2-nucleon wavefunctions possible for the deuteron.

	$\psi = \frac{u_l(r)}{r} P_{lm}(\theta) e^{im_l \phi}$	$\chi(\text{spin})$	$\tau(T)$	
1	$l=0$ symmetric	$S=1$ symmetric	$T=0$ anti-sym	deuteron <u>ANTI</u>
2	$l=0$ symmetric	$S=0$ anti-sym	$T=1 (T_z=0)$ symmetric	deuteron <u>ANTI</u>
3	$l=0$ symmetric	$S=0$ anti-sym	$T=1 (T_z=+1)$ symmetric	dineutron <u>ANTI</u>
4	$l=0$ symmetric	$S=0$ anti-sym	$T=1 (T_z=-1)$ symmetric	diproton <u>ANTI</u>

So we see that there are 4 possible combinations that are allowed for a 2-nucleon system.

1. is the bound state of the deuteron

2. is a legal state of the deuteron, but we found that this is unbound by about 60 keV. This led us to the important discovery that the nuclear central force depends on the spin orientation; the potential is shallower for anti-parallel spins.

3 and 4 are legal states, but clearly if 2 is unbound so are they.

Lecture 5

Scattering

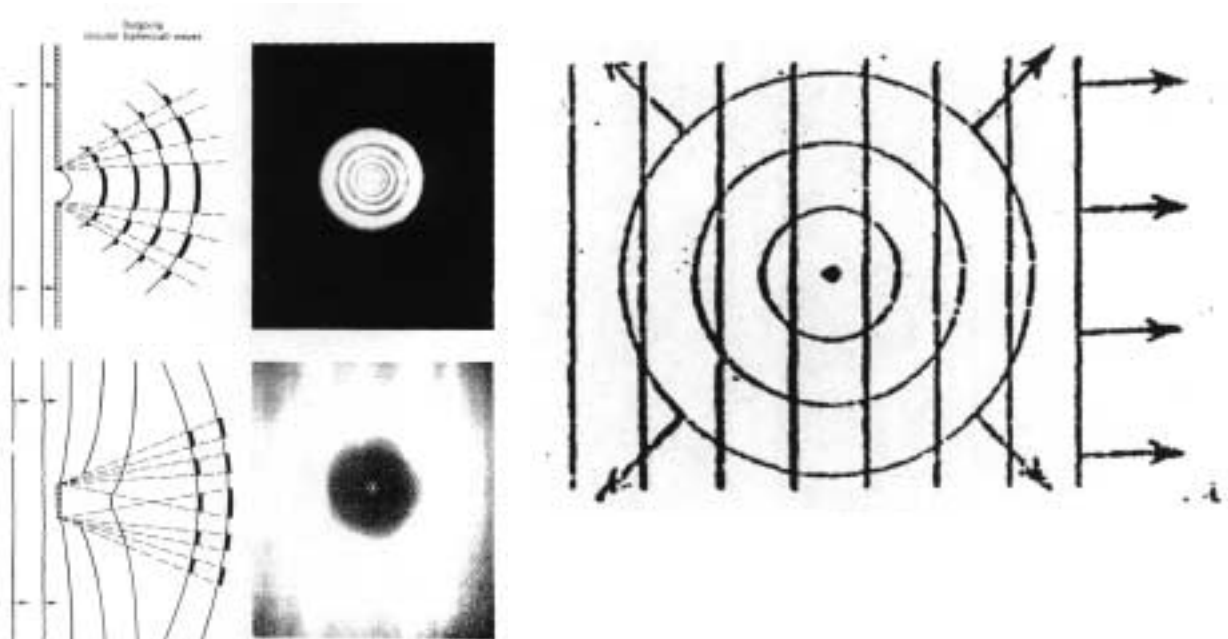
You might note that we can only find out about the ($S=1, T=0$) combination of the 2 nucleons from our study of the deuteron. To learn about the ($S=0, T=1$) couplings we need to study the effects of scattering between the nucleons. (the info about the unbound singlet ($S=0$) state that I gave you really comes from scattering data).

In studying the bound state of the 2-nucleon system (the deuteron) we have only found out the properties of the nuclear force for the case of p and n coupled to $S=1$, and for them in a state with $l = 0$. In scattering we can have nucleon pairs of all combinations (pp, pn, nn) with all possible spin alignments ($S=0, S=1$), and passing with all values of ℓ (selected by the incident energy).

We already have indications that the nuclear force is complicated. We saw that there was a tensor component that was present for $S=1$ (not of course for $S=0$). We also concluded that for $S=0$ the potential was less deep, since the $S=0$ state of the deuteron is unbound.

In scattering, the effect of nucleons coming into close proximity with the nuclear force will affect the incident wave. It produces a phase change, so that interference effects will produce a probability distribution for the scattered particles that can be analysed in terms of the potential-well depth and shape.

The analogy with optical scattering around an object is very close. The wavelength (in this case the de Broglie wavelength) and the object have to be of the same order.



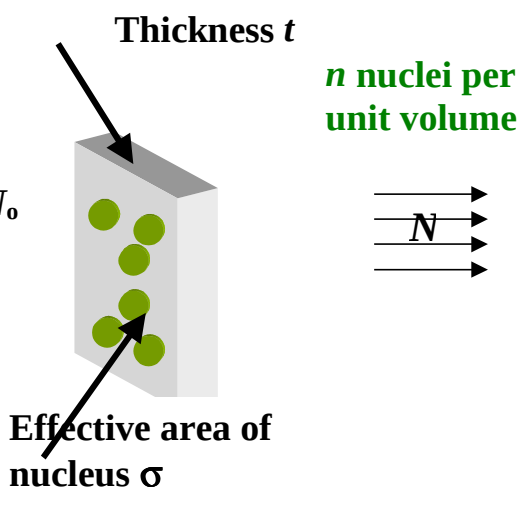
In N-N scattering, the incident particle is considered to be a plane wave with wavelength dependent on its momentum. After encountering the nuclear potential there is an outgoing spherical wave as shown. Well away from the potential we will see interference effects, which we will observe as an intensity distribution as a function of angle.

The interference is the result of a phase shift induced by the nuclear potential. The magnitude of this shift is directly relatable to the form of the potential.

How do we quantify the scattering probability? How can we categorise it as a function of say the energy of the incident particle energy? How do we categorise the scattering probability as a function of the angle of scattering? We need to have a means of comparing the data with theoretical predictions.

Hence I want now to define the term used to quantify this...**the scattering cross section.**

Reaction Cross section



The diagram illustrates a rectangular target of thickness t containing n nuclei per unit volume. Incident particles, represented by red arrows, enter from the left with an initial number N_0 . Some particles are scattered, and the remaining number N exits to the right. The effective area of a nucleus is denoted by σ .

$$-\frac{dN}{N_0} = \frac{\text{eff area of nuclei}}{\text{area of target}}$$

$$= \frac{A dx n \sigma}{A}$$

$$= n \sigma dx$$

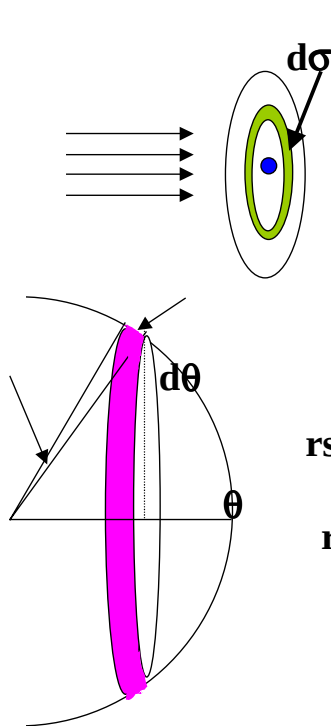
So the number scattered in thickness t is

$$-\int_{N_0}^N \frac{dN}{N_0} = n \sigma \int_0^t dx$$

$$N = N_0 (1 - e^{-n \sigma t})$$

For a thin target the number scattered is $N_0 n \sigma t$

Effective area of nucleus for scattering at angle θ is $d\sigma$.



$$\frac{-dN(\theta)}{N_o} = nt \frac{d\sigma}{d\Omega}(\theta) \Delta\Omega$$

differential cross section

$\Delta\Omega$ = solid angle between θ and $d\theta$

$$\begin{aligned} \Delta\Omega &= \frac{\text{area of ring}}{r^2} \\ &= \frac{2\pi r \sin\theta \, r \, d\theta}{r^2} \\ &= 2\pi \sin\theta \, d\theta \end{aligned}$$

$$\therefore -\frac{dN(\theta)}{N_o} = nt \frac{d\sigma}{d\Omega} 2\pi \sin\theta \, d\theta$$

$$-\int \frac{dN(\theta)}{N_o} = nt 2\pi \int_0^\pi \frac{d\sigma}{d\Omega} \sin\theta \, d\theta$$

$$\therefore \frac{dN}{N_o} = n\sigma t$$

so that

$$\sigma = 2\pi \int_0^\pi \frac{d\sigma}{d\Omega}(\theta) \sin\theta \, d\theta$$

$$\sigma = 4\pi \frac{d\sigma}{d\Omega} \quad \left| \begin{array}{l} \text{If we limit ourselves} \\ \text{to AM } l=0 \end{array} \right|$$

Nuclear Cross section

For scattering, as for all nuclear reactions, we use a characteristic called the Cross Section. Although for many situations the “cross section” is close to the physical cross section of the target nucleus, it should not be seen as that. It is like the “effective” cross sectional area of the interacting target/projectile cross section.

If life was simple and we were throwing balls at coconuts the cross section is always the same. For nucleon scattering the cross section we will see that σ varies with several variables, particularly the energy of the incident particles

The usual unit of cross section is 10^{-28} m^2 , which is approximately the physical area of a nucleus. In a kind of physicist dry humour, 10^{-28} m^2 is assigned the name “barn”. It was thought the chance of a neutron hitting a nucleus of this cross section was about as easy as hitting a barn door with a shotgun.

Scattering Experiments

What we are going to do is direct a beam of neutrons onto a target of protons, or a beam of protons onto a target of protons, and see how many of the incident particles are scattered out of the beam. The experiment is in principle very simple.

The questions we need to ask are :

- 1 How do we do it?
 - a where do we get the beam of protons of known energy from?
 - b where do we get a beam of neutrons of known energy from?
 - c where do we get a target of protons from?
 - d how do we detect the particles that do not get scattered out, or conversely how do we measure the particles that are scattered?
 - e how do we quantify the probability of scattering?
- 2 How does the scattering occur and what does it tell us about the nuclear potential? We will discuss this in some detail next lecture.

The first set of questions is the nuts and bolts of experimental nuclear physics. We need the toys to do the measurements to feed the data to the theorists.

- a Protons are of course the nuclei of hydrogen atoms. So to get protons we simply strip off the electron (usually with a combination of heating and magnetic field). Having got a charged proton it is simply a matter of putting it between a high electrical potential and voila! energetic protons.

The part-3 proton accelerator is a simple example, which provides protons with energy of 200 keV

The Pelletron in the basement is an example of a more energetic accelerator, providing protons of 3 MeV. The positive terminal in this case is charged by charge carried up on a pelletised belt. The energy is determined by the potential at the source.

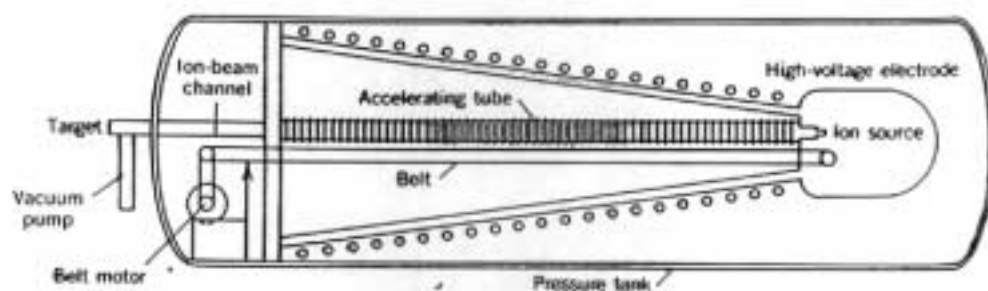


Figure 15.7 Diagram of Van de Graaff accelerator. The ion source is inside the high-voltage terminal, and both are contained in a pressure tank to inhibit sparking.

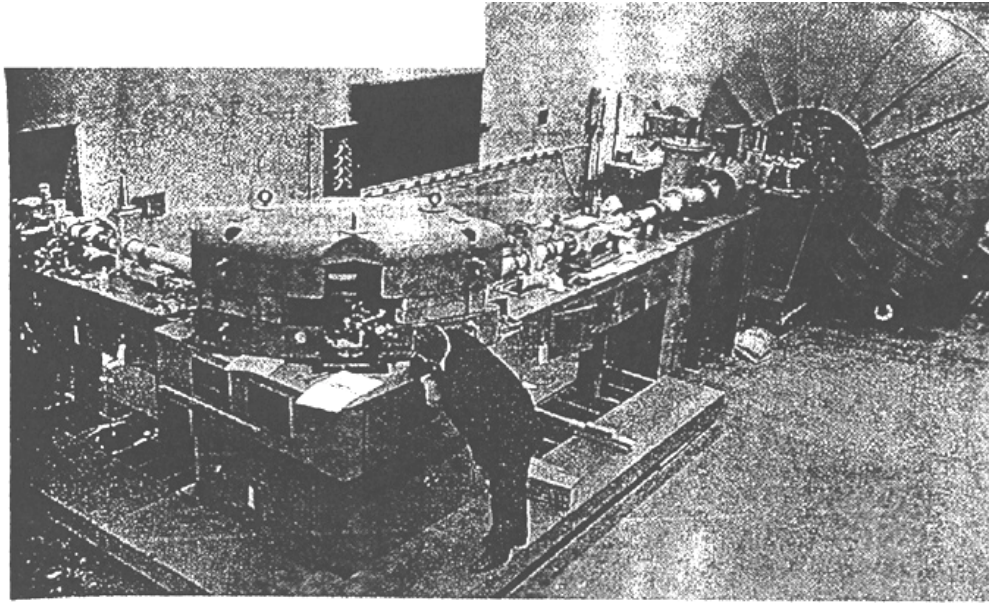


Figure 15.8 A Van de Graaff accelerator, showing the pressure tank at right, the emerging beam line, and a bending magnet to direct the beam to the experimental area. Courtesy Purdue University.

There are cyclotrons that accelerate protons by alternating HF electric fields, while confining the protons to move in a circular orbit.

572 NUCLEAR REACTIONS

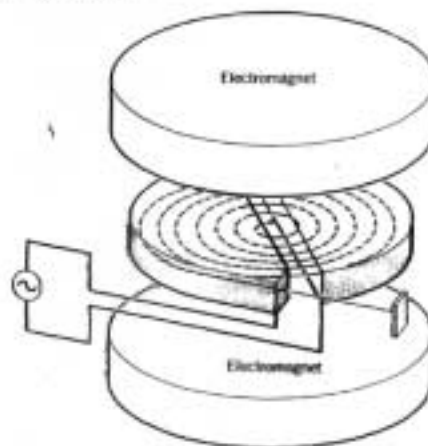


Figure 15.11 Simplified diagram of a cyclotron accelerator. The beam outward from the center, accelerated each time it crosses the gap between dees, and is eventually extracted and directed against a target.

- b You cannot accelerate neutrons, they have no charge, so if one wants a beam of neutrons they have to be produced by a nuclear reaction. The energy of the neutron is usually measured by measuring the time it takes (usually nano sec) to cover the distance from where it is produced to where it interacts.
- c You can't collect protons as such, together to make a target from which to scatter. However a target of hydrogen is ideal (either gaseous or liquid). The electrons will be unseen by any neutrons incident on the sample (why?), and if protons are being

scattered the effect of an electron on the proton with a mass 1/2000 times larger is like the effect on a truck of hitting a basketball.

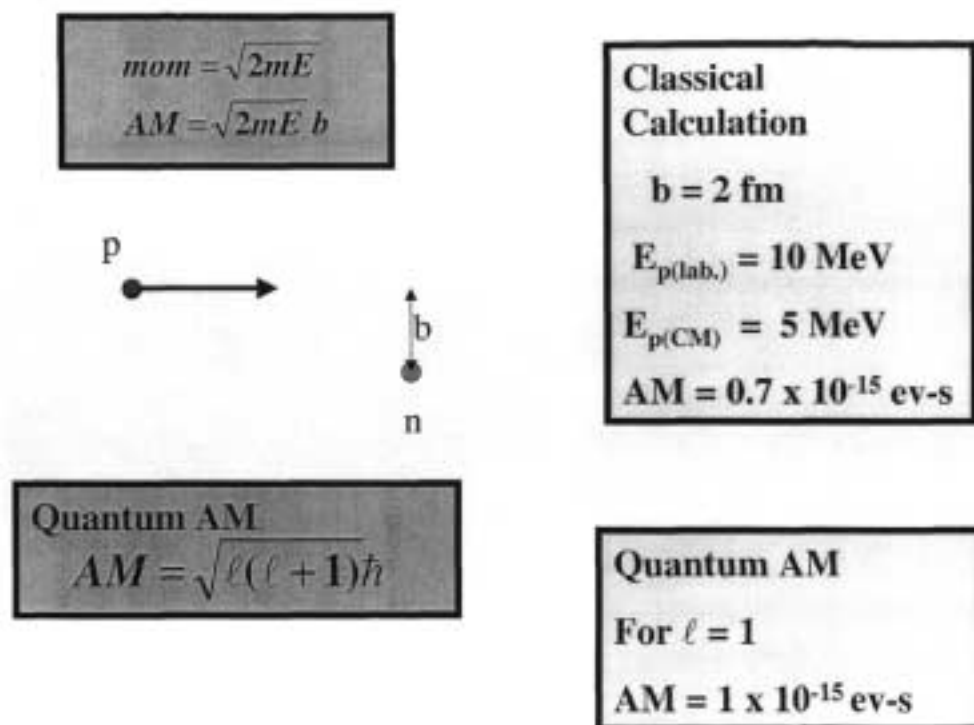
- d Detecting protons is not difficult. They ionise matter and this ionisation can be measured. In the part 3 lab, in the part-3 nuclear lab you use solid state detectors to detect α particles. They work equally well for protons. Scintillation detectors such as NaI also can be used.

$\ell=0$ Scattering

We are now in a position to see what quantum scattering theory predicts for scattering of say protons from neutrons. We could do this for other than $\ell=0$. In fact if we want to find out the full nature **of all the terms in the nuclear potential** this would have to be done. I refer you to Chap 3 of Enge or Chap 4 of Krane, if you wish to follow this further, and the notes from Eisberg and Resnik that I have given you..

You should note that the assumption of $\ell = 0$ is quite valid up to nucleon energies of a few MeV.

To see this imagine a neutron of momentum p approaching a proton at an impact parameter of b (about 1 fm. The range of the nuclear potential).



The Quantum AM is $\hbar\sqrt{\ell(\ell + 1)}$. Which if we put $\ell=1$ gives about 10^{-15} ev-sec. The classical $AM = pb$. At 10 MeV (lab) = 5 MeV (CM) the classical AM is $b.(2mE)^{1/2}$ which gives $\sim 0.6 \times 10^{-15}$ ev-s. Much less than the case for $\ell=1$. So at 10 MeV we can assume, on this semi-classical argument that the relative AM is $\ell=0$.

I want you to check this

Remember that the object of studying the scattering of p-n , p-p, is to find out the dependence of the nuclear potential on ℓ , and S. You will recall that the only combination we got from the study of the deuteron was $S=1 \ell=0$ for p and n. $S=0$ with $\ell=0$ violated the Pauli principle. We can however get all the terms from scattering experiments.

The (time independent) wavefunction of a beam of particles in free space is

$$\varphi_{in} = e^{ikz} = e^{ikr \cos \theta}$$

where $k = \frac{1}{\hbar} \sqrt{2mE}$

Note that at $r = 0$ the WF goes to zero by virtue of the repulsive core.

In fig (a) the nuc. pot. $V = 0$,

If the incoming particles gets within the range of the Nuc. Pot. The wavelength in this region will differ from k .

If the pot is attractive the wavelength will decrease, and conversly if the pot is repulsive.

In the region of the potential the value of k goes to K

What might we expect the scattering cross section to look like? First we should expect a classical diffraction pattern. However if λ is $\gg$ than the size of the scatterer, we should expect a very broad forward diffraction maximum, close to an isotropic scattering pattern. This is the case of a long wavelength sound wave being emitted from a small speaker.

Firstly lets look qualitatively at what we might expect.

The (time independent) wavefunction of a beam of particles in free space is

$$\text{where } K = \frac{1}{\hbar} \sqrt{2m(V_0 + E)}$$

This change is manifested as a phase change between the unperturbed and perturbed wavefunction, δ . So a measurement of δ will tell us the nature of the scattering potential. How can this be achieved?

The very process of scattering says that some of the particles in the incident beam will, as a result of encountering the nuc. Pot., be removed from the beam, and appear at a different angle. The first suggestions should be to measure the number of scattered particles (the scattering cross section), and relate this to the theoretical calculation.

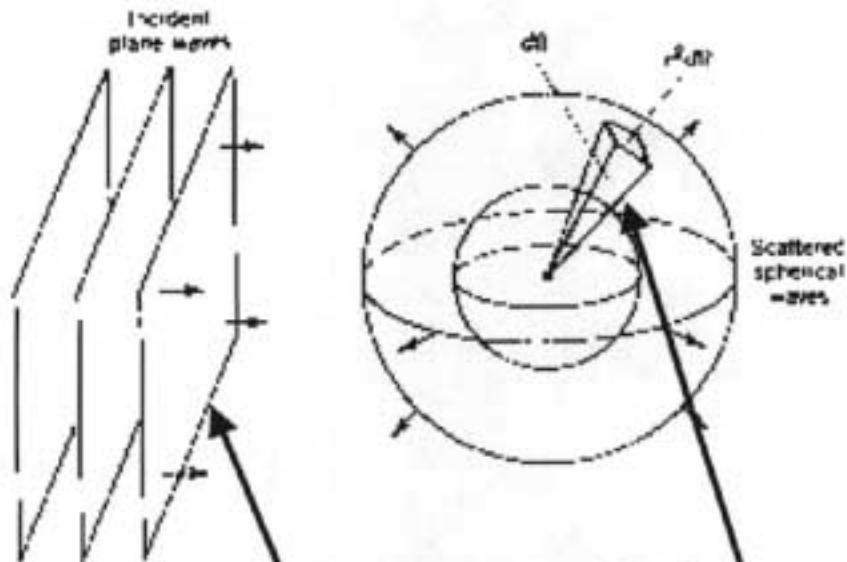


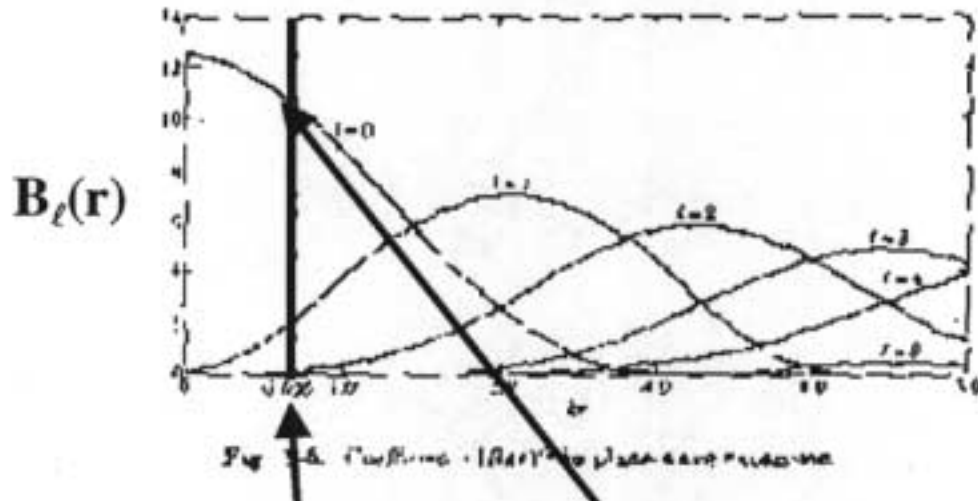
Figure 4.5 The basic geometry of scattering.

the number of particles at a given angle is the square of the amplitude of the WF, resulting from the interference of the incident beam (plane wave) and the scattered beam (spherical wave)

The interference resulting from a plane wave and a spherical wave is most easily calculated if we express the plane wave as a sum of spherical waves

$$\varphi_{in} = e^{ikz} = e^{ikr \cos \theta} = \sum_{\ell=0}^{\infty} B_{\ell}(r) Y_{\ell,0}(\theta)$$

The radial function B is a sum of spherical bessel fns and looks like Fig 3.5 (Enge).



As long as $kr < 0.7$ the $\ell=0$ contribution is by far the most important. We have seen that 2 nucleons with energy ~ 1 MeV passing within the range of the nuc. Pot. will have $\ell=0$. So let's limit our discussion to low-energy scattering. You will see that the angular distn is determined by $Y_{\ell 0}$, (fig 3.6), $\ell=0$ we assume no angular dependence $Y_{\ell m} \rightarrow (1/4\pi)^{1/2}$.

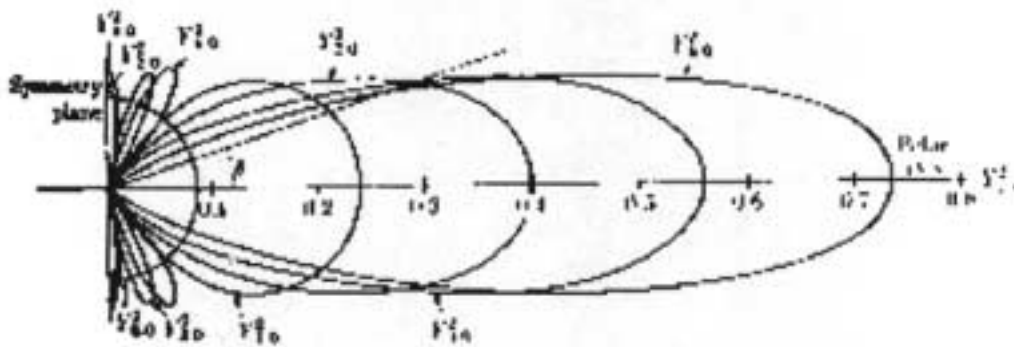


Fig. 3-6. Polar diagram of the functions $Y_{\ell 0}(\theta)$

For $\ell=0$

$$\varphi_{in} = \frac{\sin kr}{kr} = \frac{e^{ikr} - e^{-ikr}}{2ikr} \quad \text{equ 1}$$

The first term (with the time dependence added) represents a spherical wave emerging from the scattering centre. The other term is a wave converging on the centre.

The diverging wave has experienced the effect of the nuc. Pot. And has suffered a phase change which is for mathematical reasons written as 2δ .

So

$$\begin{aligned} \varphi &= \frac{e^{i(kr+2\delta)} - e^{-ikr}}{2ikr} \\ &= e^{i\delta} \frac{\sin(kr + \delta)}{kr} \end{aligned} \quad \text{equ 2}$$

Just as for equ 1, there are two terms here the first represents the outgoing wave after interaction with the nuc. pot. The other is the incoming wave. Taking the difference between equ 1 and 2 leaves only the outgoing wave, and the consequent interference.

$$\varphi = \frac{e^{ikr} - e^{-ikr}}{2ikr} - \frac{e^{i(kr+2\delta)} - e^{-ikr}}{2ikr}$$

$$\frac{e^{i(kr+2\delta)}}{kr} \sin \delta$$

This is a wave of amplitude $\frac{\sin \delta}{kr}$ moving away from the scattering centre.

The number of particles with vel v , carried by this wave per sec. is got by integrating over the sphere:

$$N_{sc} = \frac{4\pi \sin^2 \delta}{k^2} v$$

the scattering cross section is the flux of scattered particles divided by the incident flux, so that

$$\sigma = \frac{4\pi \sin^2 \delta}{k^2}$$

Note again that for $\ell = 0$ there is no angular dependence.

The theoretical cross section is obtained by finding δ .

This value is obtained by joining the wave function of the scattered wave back at the potential.

For $r > c$ we have $\psi = B \sin(kr + \delta)$, and for $b < r < c$ we have $\psi = A \sin Kr$

Matching the magnitude and the derivatives gives the transcendental equation

$$K \cot Kb = k \cot [k(c+b) + \delta]$$

Using the values that we used in solving the deuteron

$V_0 = 73$ MeV (determines K for a given E)

$b = 1.34$ fm

$c = 0.4$ fm

gives values for δ as a function of E .

The theoretical value is about 5 mb. The result for the measured cross section is shown

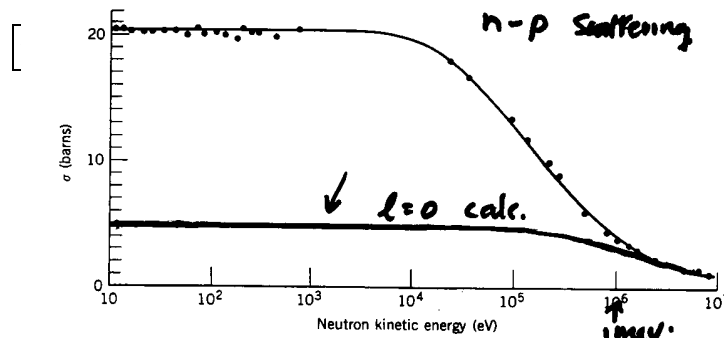


Figure 4.6 The neutron-proton scattering cross section at low energy. Data taken from a review by R. K. Adair, *Rev. Mod. Phys.* **22**, 249 (1950), with additional recent results from T. L. Houk, *Phys. Rev. C* **3**, 1886 (1970).

We see that the theoretical value is significantly below the measured value at low energies. Why is this? And what does it tell us?

We did the calculation using the parameters that gave the bound deuteron. This arrangement of n-p was parallel spins for n and p:

$S=1$. However in scattering we are not limited to $S=1$, we also have $S=0$. As we noted before, the nuclear potential for $S=0$ is smaller than for $S=1$. Recall the S^1 state of the deuteron was unbound. If we assume that the large difference is due to singlet scattering we can determine the relative strengths of σ_3 and σ_1 .

As the neutron, with spin $\frac{1}{2}$ approaches the proton in the target (with spin $s = \frac{1}{2}$), the probability of coupling to 1 is 3 times that of coupling to 0.

Thus the measured cross section is

$$\sigma = \frac{3}{4} \sigma_3 + \frac{1}{4} \sigma_1.$$

We have calculated σ_3 to be about 5 b, since we used $S=1$ parameters.

$$\text{So } 20 \text{ b} = \frac{3}{4} (5 \text{ b}) + \frac{1}{4} \sigma_1$$

$$\frac{1}{4} \sigma_1 = 20 - \sim 4 = 16 \text{ b}$$

$$\text{So that } \sigma_1 \sim 64 \text{ b.}$$

This is a huge difference, and an important indication of the difference in the singlet and triplet components of the nuclear potentials. This confirms our observation from the fact that the S^1 excited state was less bound than the S^3 , that the nuclear force must be spin dependent.

It would be unfair of me to suggest that we have really covered N-N scattering. Most of the information about the nature of the N-N force was obtained from more complicated scattering than $\ell=0$. I do not intend to cover that. You will find it in Enge Chap. 3 and Krane Chap. 4.

At higher values of ℓ the interference pattern becomes more complex, and the analysis more difficult (and more subjective).

Suffice it to say that at higher energies the ℓ -s components can be found, and as the energy gets higher the nucleon probes closer to the repulsive core, and this can be resolved.

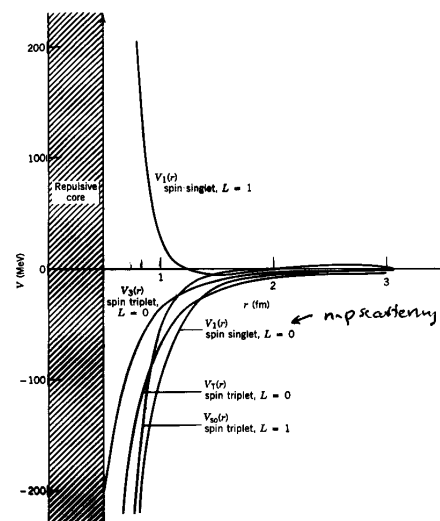


Figure 4.10 Some representative nucleon-nucleon potentials. Those shown include the attractive singlet and triplet terms that contribute to s-wave scattering, the repulsive term that gives one type of p-wave ($L=1$) scattering, and the attractive tensor and spin-orbit terms. All potentials have a repulsive core at $r = 0.49$ fm. These curves are based on an early set of functional forms proposed by T. Hamada and I. D. Johnston, *Nucl. Phys.* **34**, 382 (1962); other relatively similar forms are in current use.