5 Phase Shifts

A plane wave incident on a target has the wavefunction:

$$\Psi_i = e^{ikz}$$

Where k = $\frac{1}{\bar{\lambda}}$, $\bar{\lambda}$ = de Broglie Wavelength =$\frac{\hbar}{p}$. We can write a plane wave as a superposition of spherical waves, which for kr $\gg$ 1 gives:

$$\Psi_i = e^{ikz} = \frac{i}{2kr}\sum_l (2l+1)\left[(-1)^l e^{ikr} - e^{ikr}\right] P_l(\cos\theta) \tag{1}$$

Where e^{-ikr} represents the incomming wave and e^{-ikr} represents the outgoing wave. Of course P$_l(\cos\theta)$ = Legendre Polynomials.

Now the scattering center cannot effect the incomming wave but it can alter both the amplitude and phase of the outgoing wave.

Let $2\delta_l$ = change in phase of the l-th partial wave.
η_l = amplitude after scattering of the outgoing wave, then $0 \leq \eta_l \leq 1$
The total wave now has the form:

$$\Psi_{tot} = \frac{i}{2kr}\sum_l (2l+1)\left[(-1)^l e^{-ikr} - \eta_l e^{ikr} e^{2i\delta_l}\right] P_l(\cos\theta) \tag{2}$$

So the scattered wave is:

$$\begin{aligned}\Psi_{scatt} = \Psi_{tot} - \Psi_i &= \frac{e^{ikr}}{kr}\sum_l (2l+1)\frac{\eta_l e^{2i\delta_l} - 1}{2i} P_l(\cos\theta) && (3)\\ &= \frac{e^{ikr}}{r} F(\theta) && (4)\\ & && (5)\end{aligned}$$

where the scattering amplitude is given by:

$$F(\theta) = \frac{1}{k}\sum_l (2l+1)\frac{\eta_l e^{2i\delta_l} - 1}{2i} P_l(\cos\theta) \tag{6}$$

This is an elastically scattering wave in the C.M. frame because the incident and outgoing k's are the same.

OK The scattered outgoing flux through a solid amgle dΩ at radius r is given by

$$\begin{aligned} v_0 \Psi_{scatt}\Psi^*_{scatt} r^2 d\Omega &= v_0 \mid F(\theta) \mid^2 \frac{e^{ikr}}{r^2} r^2 d\Omega && (7)\\ &= v_0 \mid F(\theta) \mid^2 d\Omega && (8)\end{aligned}$$

Where v_0 = velocity of the outing wave
$\Psi_{scatt}\Psi^*_{scatt}$ = Probability density
$r^2 d\Omega$ = surface area at r
Now the cross section dσ is given by

$$d\sigma = \Psi_{scatt}\Psi^*_{scatt} r^2 d\Omega \tag{9}$$

$$\Rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{el} = \mid F(\theta) \mid^2 \tag{10}$$

Therefore the total elastic cross section is

$$\begin{aligned} \sigma_{el} = \int_{4\pi} \frac{d\sigma}{d\Omega} d\Omega &= \int \mid F(\theta) \mid^2 d\Omega \\ &= \int \frac{1}{k^2} \left(\sum_l (2l+1) \frac{\eta_l e^{2i\delta_l} - 1}{2i} P_l(\cos\theta) \right) \\ &\times \left(\sum_{l'} (2l+1) \frac{\eta_l' e^{2i\delta_l} - 1}{2i} P_l'(\cos\theta) \right) d\Omega \end{aligned} \tag{11}$$

However

$$\int P_l(\cos\theta) P_{l'}(\cos\theta) d\Omega = \frac{4\pi\delta_{ll'}}{2l+1}$$

$$\Rightarrow \quad \sigma_{el} = \frac{1}{k^2} \sum_l \frac{(2l+1)^2 4\pi}{2l+1} \mid \frac{\eta_l e^{2i\delta_l} - 1}{2i} \mid^2 \tag{12}$$

$$= 4\pi\bar{\lambda}^2 \sum_l (2l+1) \mid \frac{\eta_l e^{2i\delta_l}}{2i} \mid^2 \tag{13}$$

When $\eta_l = 1$, ie no absorption of the incomming wave,

$$\sigma_{el} = 4\pi\bar{\lambda}^2 \sum_l (2l+1) \sin^2 \delta_l$$

$\Rightarrow$ $\sigma_{el} = 0$ when $\delta_l = 0$. ie No scattering potential $\Rightarrow$ no phase shift $\Rightarrow$ no cross section.

Now if $\eta_l < 1$ we get the reaction cross section, σ_r from conservation of probability.

$$\begin{aligned} \sigma_r &= \int \left(\mid \Psi_{in}^2 - \mid \Psi_{out} \mid^2 \right) r^2 d\Omega \\ &= \pi\bar{\lambda}^2 \sum_l (2l+1)(1-\eta_l^2) \end{aligned} \tag{14}$$

The total cross section is

$$\sigma_T = \sigma_r + \sigma_{el} = \pi\bar{\lambda}^2 \sum_l (2l + 1_2(1 - \eta_l \cos(2\delta_l)) \tag{15}$$

Now since $P_l(1) = 1$, for all l and $\cos\theta = 1 \Rightarrow$

$$\begin{aligned}
F(\theta) &= \frac{1}{k}\sum_l (2l+1)\frac{\eta_l e^{2i\delta_l} - 1}{2i} \\
&= \frac{1}{k}\sum_l (2l+1)\left(\frac{\eta_l(\cos(2\delta_l) + i\sin(2\delta_l)) - 1}{2i}\right) \\
&= \frac{1}{2k}\sum_l (2l+1)(\eta_l(\sin(2\delta_l) - i\cos(2\delta_l)) + i) \\
\Rightarrow \quad ImF(0) &= \frac{1}{2k}\sum_l (2l+1)(1 - \eta_l\cos(2\delta_l)) \\
&= \frac{k}{4\pi}\left(2\bar{\lambda}^2\pi\sum_l (2l+1)(1 - \eta_l\cos(2\delta_l))\right) \\
&= \frac{k}{4\pi}\sigma_T \qquad (16)
\end{aligned}$$

this is the optical theorem which relates the imaginary part of the forward amplitude to the total cross section.

let's get the maximum cross sections:

$$\sigma_{el}^{max} = 4\pi\bar{\lambda}^2(2l+1) \quad ie.when \quad \eta_l = 1$$

$$\sigma_{r}^{max} = \pi\bar{\lambda}^2(2l+1) \quad ie.when \quad \eta_l = 0$$

let's look at some simple classical arguments as to what the reaction cross section mean.

Let b= impact parameter = distance of closest approach to target. Then $p \times b = l\,\hbar$ where l = orbital angular momentum quantum number and the product $p \times b$ is the classical angular momentum.

$$\Rightarrow \quad b = \frac{l\hbar}{p} = l\bar{\lambda} \qquad (17)$$

Where $\bar{\lambda}$ is the de Broglie wavelength. This gives the particle "size". So the cross sectional area between l and l+1 is given by

$$\sigma_l = \pi(b_{l+1}^2 - b_l^2) = \pi\bar{\lambda}^2(2l+1)$$

This increase in cross section with l is always balanced by a reduction in the scattering potential or phase shift. It takes a lot of momentum to induce angular momentum. Look at the quantity.

$$f(l) = \frac{\eta_l e^{2i\delta_l} - 1}{2i} = \frac{i}{2} - \frac{i\eta_l}{2}e^{i2\delta_l} \qquad (18)$$

This is the amplitude of lth partial wave. As can be seen f(l) = 0 when $\delta_l = 0$ and is a maximum when $\delta_l = \frac{\pi}{2}$. When $\delta_l = \frac{\pi}{2}$ the scattering amplitude goes through a resonance.

5.1 The Breit-Wigner Resonance Formula

We can rewite f(l) as

$$\begin{aligned} f(l) &= \frac{e^{i\delta_l}(e^{i\delta_l} - e^{-i\delta_l})}{2i} \\ &= e^{i\delta_l} sin\delta_l \\ &= \sin\delta_l \cos\delta_l + i\sin^2\delta_l \\ &= \sin^2\delta_l(\cot(\delta_l + i)) \\ &= \frac{\sin^2\delta_l(\cot^2\delta_l + 1)}{\cot 2\delta_l - i} \\ &= \frac{\cos^2\delta_l + \sin^2\delta_l}{\cot\delta_l - i} \\ &= \frac{1}{\cot\delta_l - i} \end{aligned} \tag{19}$$

Now near resonance $\delta_l \approx \frac{\pi}{2} \Rightarrow \cot\delta_l \approx 0$ so let's expand $\cot\delta_l$ in a Taylor series:

$$\begin{aligned} \cot\delta_l(E) &= \cot\delta_r(E_R) + (E - E_R)\left[\frac{d}{dE}\cot\delta_l(E)\right]_{E=E_R} + ... \\ &\approx -(E - E_R)\frac{2}{\Gamma} \end{aligned} \tag{20}$$

Since $\cot\delta_{E_R} = 0$ and we defined:

$$\frac{2}{\Gamma} = -\left[\frac{d}{dE}\cot\delta_l(E)\right]_{E=E_R} \tag{21}$$

We can neglect further terms provided $\mid E - E_R \mid \approx \Gamma \ll E_R$. If the resonance is broad and near threshold then the resonance is assymetric.

OK, now substituting for $\cot\delta_l$ with the above approximation gives:

$$f(E) = \frac{1}{\cot\delta_l - i} = \frac{1}{-(E - E_R)\frac{2}{\Gamma} - i} = \frac{\frac{\Gamma}{2}}{((E_R - E) - \frac{i\Gamma}{2})} \tag{22}$$

Then the cross section for elatic scattering is:

$$\sigma_{el}^l(E) = 4\pi\bar{\lambda}^2(2l + 1)\frac{\frac{\Gamma}{4}^2}{(E - E_R)^2 + \frac{\Gamma}{4}^2} \tag{23}$$

This is the Breit-Wigner Resoance formula which maps out an increase in cross section as the phase shifts pass through 90 degrees. Γ is defined so to equal the width of the resoance curve at half height. Γ is called the resoance width and is related to the resoance lifetime via the uncertainty principle.

$$\Gamma\tau = \hbar \quad \Rightarrow \quad \tau = \frac{\hbar}{\Gamma} = lifetime\ of\ the\ state \tag{24}$$

For a spinless particle:

$$\sigma_{el} = 4\pi\bar{\lambda}^2(2l+1)\frac{\frac{\Gamma}{4}^2}{(E-E_R)^2+\frac{\Gamma}{4}^2} \tag{25}$$

However, in general, for two particles with spins to produce a particle C of spin J:

$$a + b \rightarrow C \rightarrow a + b$$

ie The particles a and b collide to form particle C which then decays back to particles a and b. This is equivalent to elastic scattering. Then:

$$\sigma_{el}(E) = \frac{4\pi\bar{\lambda}^2(2J+1)\frac{\Gamma}{4}^2}{(2S_a+1)(2S_b+1)\left[(E-E_R)^2+\frac{\Gamma}{4}^2\right]} \tag{26}$$

In addition to eleastic scattering it is also possible for reactions to occur. ie Resoance C decays into particles that are not the same as the incident particles.

eg.

$$\begin{aligned} \pi + n \rightarrow \quad & \Delta \quad \rightarrow \pi + n \\ also \quad & \Delta \quad \rightarrow \pi\pi n \end{aligned}$$

In general, $\Gamma = \Gamma_{el} + \Gamma_{reaction}$, where Γ_{el} and Γ_{re} are "partial widths" for the decay into the elastic and ineleastic channels.

Then for $a + b \rightarrow C \rightarrow d + e$:

$$\sigma_{ab\rightarrow de}(E) = \frac{4\pi\bar{\lambda}^2}{(2S_a+1)(2S_b+1)}\frac{\Gamma_{ab}\Gamma_{de}}{\left[(E-E_R)^2+\frac{\Gamma}{4}^2\right]} \tag{27}$$

Where $\Gamma = \Gamma_{el}$ + all reaction widths. Γ_{el} = width for the decay C to a and B.

5.2 Example of a resonance. $e^+ + e^- \rightarrow Z^0$

The full width of the Z^0 resonance is:

$$\Gamma = \Gamma_{Z\rightarrow e^+e^-} + \Gamma_{Z\rightarrow\mu^+\mu^-} + \Gamma_{Z\rightarrow\tau^+\tau^-} + \Gamma_{Z\rightarrow\nu_e\bar{\nu}_e} + \Gamma_{Z\rightarrow\nu_\mu\bar{\nu}_\mu} + \Gamma_{Z\rightarrow\nu_\tau\bar{\nu}_\tau} + \Gamma_{Z\rightarrow hadrons} \quad (28)$$

So for example:

$$\sigma_{e^+e^-\rightarrow Z^0\rightarrow\mu^+\mu^-}(E) = \frac{4\pi\bar{\lambda}^2}{(2S_e+1)(2S_e+1)}\frac{\Gamma_{e^+e^-}\Gamma_{\mu^+\mu^-}}{((E_R-E)^2+\frac{\Gamma}{4}^2)} \quad (29)$$

So we can measure the cross sections for $e^+e^- \rightarrow$ all these processes except $Z\rightarrow \nu\bar{\nu}$'s

However we can measure the total width from the "width" of the resonance. Then the difference between the full width Γ and the the sum of the visible partial widths $= \sum_i \Gamma_{\nu_i\bar{\nu}_i}$ ie

$$\sum_i \Gamma_{\nu_i\bar{\nu}_i} = \Gamma - \sum_i \Gamma_i \quad (30)$$

Now each partial width :

$$\Gamma_{\nu_e\bar{\nu}_e} = \Gamma_{\nu_\mu\bar{\nu}_\mu} = \Gamma_{\nu_\tau\bar{\nu}_\tau} = \Gamma_{\nu_x\bar{\nu}_x} = 168MeV$$

so if there is an extra neutrino and hence another generation of quarks and if $m_{\nu_{tau}} <$ 45 GeV then it would show up in the "invisible" width of the Z^0 decay.

In fact $\Gamma_{\nu\bar{\nu}}$ = 489.3 ± 4.2 MeV

$$\Rightarrow \quad N_\nu = \frac{\Gamma_{\nu\bar{\nu}}}{168} = 3.03 \pm 0.04$$

Which is an excellent test of the Standard Model and strong evidence of the existance of just 3 generations of quarks and leptons.

6 Quark model of Hadron Structure

By the early 1960's a plethora odf new particles had been discovered. There had also been a number of strange particles observed. When laid out as a function of strangeness and third component of isospin, a regular arrangement was observed.

For particles of spin $\frac{3}{2}$.

Strangeness	-$\frac{3}{2}$	-1	-$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	mass	I$_{tot}$
s= 0	Δ^-		Δ^0		Δ^-		Δ^+	1232	$\frac{3}{2}$
s= -1		σ^{*-}		σ^{*0}		σ^{*+}		1384	1
s= -2			Ξ^{*-}		Ξ^{*-}			1583	$\frac{1}{2}$
s= -3				ω^-					0

The pattern shown above did not originally have the ω^- in place. It's existance was deduced from the table of elements of the other spin 3/2 particles. This in turn led to the notion of quarks by Gell-Mann. After the prediction an intensive search was mounted at Brookhaven National Laboratory to search for the particle. It's observation in 1963 led to a Nobel Prize for the discoverers two years later.

All these particles, except the Ω^-, decay strongly to other baryons. eg.

7 Neutral Kaons

8 The Weak Interaction in More detail

The weak