Lock-in amplifiers

A short tutorial by R. Scholten
Measuring something

- Common task: measure light intensity, e.g. absorption spectrum
- Need very low intensity to reduce broadening
- Noise becomes a problem
The principle

- Fundamental law of communication theory: **Wiener-Khinchin** theorem
- **Reduction** of noise imposed upon a useful signal with frequency $f_0$, is proportional to the **square root of the bandwidth** of a bandpass filter, centre frequency $f_0$
Typical photodetector noise spectrum

- Bad noise here...
- Off resonance
- Dark noise
- 90Hz $\Delta f$
- On resonance

Better here!

...and here!

**dc measurements:**
- broad-spectrum (bad)
- at low frequency (bad)
Measuring something

- Need to measure at high frequency, where noise is low
- Modulate signal, look for component oscillating at modulation frequency

![Diagram of Rb cell, photodiode, laser, and chopper with lock-in amplifier](image)

![Rb spectrum](image)
A lock-in amplifier is used to extract signal from noise. It detects signal based on modulation at some known frequency.

Premise:
- much noise at low frequency (e.g. dc), less noise at high frequency
- measure within narrow spectral range, reduce noise bandwidth

Hence shift measurement to high frequency

Figures from Bentham Instruments document 225.02 Lockin amplifiers
Demodulator or PSD (phase-sensitive detector)
Mathematical description

- Signal $V_S(t)$ varies relatively slowly
  e.g. absorption spectrum scan over 10 seconds

- Modulate at relatively high frequency $\omega$ (e.g. chopper):
  
  $V_{\text{sig}} = V_S(t)\cos \omega t$

- Reference (local oscillator) of fixed amplitude:
  
  $V_{\text{ref}} = \cos(\omega t + \phi)$

- phase $\phi$ is variable
- oscillator frequency $\omega$ same as modulation frequency

- Multiply modulated signal by REF:
  
  $V_{\text{sig}} V_{\text{ref}} = V_S(t)\cos \omega t \cos(\omega t + \phi)$
  
  $= \frac{1}{2} V_S(t)\cos \phi + \frac{1}{2} V_S(t)\cos(2\omega t + \phi)$

- Second term at high frequency ($2\omega$)

- Low-pass filter (cutoff $\sim \omega/2$ or lower)

  $V_{\text{sig}} \times V_{\text{ref}} \times \text{filter} = \frac{1}{2} V_S(t)\cos \phi$

Note phase-sensitive detection!
Simple trig

\[ V_{\text{ref}} V_{\text{sig}} = V_s(t) \cos \omega t \cos(\omega t + \phi) + n(t) \cos(\omega t + \phi) \]

\[ = V_s(t) \cos \omega t [\cos \omega t \cos \phi - \sin \omega t \sin \phi] + \ldots \]

\[ = V_s(t) \cos \omega t [\cos \omega t \cos \phi - \cos \omega t \sin \omega t \sin \phi] + \ldots \]

\[ = V_s(t) \left[(\frac{1}{2} + \frac{1}{2} \cos 2\omega t) \cos \phi - \frac{1}{2} \sin 2\omega t \sin \phi\right] + \ldots \]

\[ = \frac{1}{2} V_s(t) \cos \phi + \frac{1}{2} V_s(t) \left[\cos 2\omega t \cos \phi - \sin 2\omega t \sin \phi\right]+\ldots \]

\[ = \frac{1}{2} V_s(t) \cos \phi + \frac{1}{2} V_s(t) \cos(2\omega t + \phi) + n(t) \cos(\omega t + \phi) \]
Noise

- Noise reduces with frequency (1/f noise is major problem)
- Shift signal to higher frequency
- Noise within given bandwidth reduces as we measure at higher frequency
With noise

- Signal has noise: \[ V_{\text{sig}} = V_{S}(t) \cos \omega t + n(t) \]

- Multiply reference by modulated signal:
  \[
  V_{\text{ref}} V_{\text{sig}} = V_{S}(t) \cos \omega t \cos(\omega t + \phi) + n(t) \cos(\omega t + \phi)
  = \frac{1}{2} V_{S}(t) \cos \phi + \frac{1}{2} V_{S}(t) \cos(2\omega t + \phi) + n(t) \cos(\omega t + \phi)
  \]

- Third term – noise – at frequency \( \omega \)
- Low-pass filter, frequency less than \( \omega/2 \), leaves signal components
- We win twice:
  - less noise at \( \omega \)
  - reduce bandwidth
Using PSD oscillator to modulate
External modulator: true “lock-in”
Further details

- True lock-in amp can work with external oscillator for Reference:
  - Input reference from external experiment
  - Use phase-locked-loop to generate stable local oscillator

- Lock-in amp has variable post-multiplier (low-pass) filter
  - Time constants: what time constant is appropriate?
  - Shapes (6\textsuperscript{th}, 12\textsuperscript{th}, … order): which is best?

- If input signal has \textit{harmonics} (e.g. due to imperfect modulation) then will detect spurious signal
  - Use \textit{input} filter to minimise

- Dynamic reserve?
Other applications

- Often use lockin to measure response function of actuator (or similar)
- Two-channel lockin – measure signal and *phase*
- Phase → resonances
Experiments

- Photodiode + LED
- SRS FFT spectrum analyser
- Oscilloscope
- Switch LED on/off, e.g. with hand to block
- HP function generator to modulate LED
- And/or chopper
- SRS lock-in amp
The other half of the story

Frequency modulation
Derivatives: Lock-in amps & feedback servos

- So far, we have modulated amplitude, and used LIA to demodulate.
- PSD (lockin) = fancy bandpass filter?!

- Can also use frequency modulation (like FM radio)

- Let’s measure $V_S(\omega)$ i.e. a spectrum, where we slowly vary $\omega(t)$

- Frequency-modulate: $\omega(t) = \omega_0 + \Omega_0 \cos \Omega t$
  where $\Omega$ is the modulation (Fourier) frequency

- Using Taylor-series expansion: 
  $$ V_S(\omega) = V_S(\omega_0) + \left. \frac{dV_S}{d\omega} \right|_{\omega=\omega_0} (\Omega_0 \cos \Omega t) + ... $$

- Note two things immediately:
  - dc component is same as un-modulated spectrum
  - ac component is proportional to derivative of spectrum
We now multiply our signal by our reference, as before:

\[ V_{\text{sig}} = V_S(\omega_0) + \left. \frac{dV_S}{d\omega} \right|_{\omega=\omega_0} (\Omega_0 \cos \Omega t) + \ldots \]

\[ V_{\text{ref}} = \cos(\Omega t + \phi) \]

Note modulation at \( \Omega \), and fixed \( \omega_0 \) i.e. slowly varying laser frequency

\[ V_{\text{sig}} V_{\text{ref}} = V_S \cos(\Omega t + \phi) + \left. \frac{dV_S}{d\omega} \right| \Omega_0 \cos \Omega t \cos(\Omega t + \phi) \]

\[ = \ldots \]

\[ = V_S \cos(\Omega t + \phi) + \frac{1}{2} \Omega_0 \left. \frac{dV_S}{d\omega} \right| \cos(2\Omega t + \phi) + \frac{1}{2} \Omega_0 \left. \frac{dV_S}{d\omega} \right| \cos \phi \]

Again: low-pass filter (cutoff \( \sim \omega/2 \) or lower)

\[ V_{\text{sig}} V_{\text{ref}} \approx \frac{1}{2} \Omega_0 \left. \frac{dV_S}{d\omega} \right| \cos \phi \]

We have a measurement proportional to the derivative

Measurement changes sign if slope changes sign: dispersion

Note: modulation depth \( \Omega_0 \) must not be larger than peak in spectrum!

Higher-order terms in Taylor expansion: can measure 2\textsuperscript{nd} deriv, 3\textsuperscript{rd} deriv, etc.
Lock-in amplifiers and feedback servos

- Example: Lorentzian peak in atomic absorption spectrum

- Smaller slope = smaller signal
- On centre:
  - small
  - double frequency

Note opposite phase: \( \cos \phi < 0 \)
Lock-in amps in servos

- Modulated output from detector

- Demodulated output from lock-in

\[ V_{\text{sig}} V_{\text{ref}} \approx \frac{1}{2} \Omega_0 \frac{dV_s}{d\omega} \cos \phi \]
Yet other half of the story

Spread-spectrum
PRBS: Lock-in on steroids!

- Lock-in uses small part of spectrum
- Can use broad spectrum and still separate signal from noise
- Pseudo-random bit sequence
  - Spread-spectrum communications
    - computer 802.11 wireless, etc.
    - CDMA telephones
    - Modems
  - Security/encryption
  - Acoustics
Spread-spectrum modulation

SPREAD-SPECTRUM MODULATION

• all frequencies present simultaneously in modulation function
• phases adjusted so that components add in quadrature

http://www.chm.bris.ac.uk/pt/mcinet/sum_schl_02_docs/tof.ppt

• truly random phases cause excursions out of range
  ⇒ use pseudo-random functions

http://www.chm.bris.ac.uk/pt/mcinet/sum_schl_02_docs/tof.ppt
Hedy Lamarr (1913-2000), composer George Antheil (1900-1959) patented submarine communication device

- Synchronized frequency hopping to evade jamming
- Original mechanical action based upon pianolas
- Used today in GPS, cellphones, digital radio

http://www.ncafe.com/chris/pat2/index.html
http://www.chm.bris.ac.uk/pt/mcinet/sum_schl_02_docs/tof.ppt

Also famous as first nude in cinema-release movie!
Binary pseudo-random sequences

- sequence length $2^n - 1$ bits with $n$ bit shift register

Autocorrelation:

$\pm 1/(2^n - 1)$

[Diagram of a clock input and shift register]
**PRBS: Lock-in on steroids!**

- Generate signal in pseudo-random bit sequence, for example:
  - 6-bit (64-bits long)
    - 0110101111110000110001010011110100011100100101101110110
  - 8-bit (256 bits long):
    - 00011011011100100101101011101101110101110110111101111110101100101000111110000010110
    - 0110010001110100010011100111011110111011111100001100110001000101001011000111001001
    - 00111100010000011111111010001010101110010111111001011111001100000101101000101101110

- Record signal
- Multiply by PRBS (auto-correlate)
- Very much like a lock-in! But uses broad spectrum
13-bit (8192 bits long) MLS single scan

Assumed velocity distribution

Detected signal

Recovered TOF

Recovered velocity distn