Nonequilibrium Statistical Mechanics (Honours)
Assignment 1

1. **Brownian pendulum.** Consider a one dimensional, simple harmonic oscillator with natural angular frequency $(\gamma^2 + \omega_0^2)^{1/2}$, subjected to a frictional acceleration $-2\gamma x'(t)$, where $x(t)$ is the oscillation amplitude, and pumped by a stochastic force per unit mass $\xi(t)$ with white noise statistics, i.e. $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = D\delta(t' - t)$. An example of such a system is a frictional pendulum bombarded by molecules in a jar of air.

   (a) Write down the equation of motion for the pendulum.

   (b) Solve explicitly for $x(t)$ given $x(0) = x'(0) = 0$ initially.

   (c) Explain physically why the initial conditions in (b) lead to a trivial result in the absence of a Langevin source but not in its presence.

   (d) Calculate $\langle v(t)^2 \rangle$ as $t \to \infty$ for an ensemble of identically prepared pendula.

   (e) Suppose the pendulum is in thermal equilibrium with a gas held at a temperature T. Deduce the fluctuation-dissipation theorem from your answer to (d).

   (f) Can you suggest an alternative approach to this problem which does not entail solving explicitly for $x(t)$ as in (b)? It is *not* necessary to attempt the calculation; just outline your suggestion briefly in words.

   Mathematical hint for (b). You may wish to use the following result obtained by the method of variation of parameters (Zwillinger 1997, §95). The particular solution of the ordinary differential equation $x''(t) + P(t)x'(t) + Q(t)x(t) = R(t)$ is given by

   $$x(t) = -x_1(t) \int dt\, x_2(t)R(t)W(x_1, x_2)^{-1} + x_2(t) \int dt\, x_1(t)R(t)W(x_1, x_2)^{-1} , \quad (1)$$

   where $W(x_1, x_2) = x_1 x_2' - x_1' x_2$ is the Wronskian of the complementary solutions $x_1(t)$ and $x_2(t)$.

   Mathematical hint for (d). You may find the following integrals useful.

   $$\int dx\ \exp(ax)\cos(bx) = (a^2 + b^2)^{-1} \exp(ax)[a\cos(bx) + b\sin(bx)] \quad (2)$$

   $$\int dx\ \exp(ax)\sin(bx) = (a^2 + b^2)^{-1} \exp(ax)[-b\cos(bx) + a\sin(bx)] \quad (3)$$

   Feel free to use *Mathematica* to do the calculations here if you don't feel like slogging through the algebra. Include a printout of the notebook (without evaluations to keep it short) with your answer if you decide to take this route.