Nonequilibrium Statistical Mechanics (Honours)
Assignment 2

1. **Singles in a jazz bar.** At night, the groovy laneways of a cosmopolitan metropolis transform themselves from placid thoroughfares into seething laboratories of the human psyche, a dimly lit netherworld where stochastic social experiments (some Markovian, some Pavlovian) are compulsively conducted as if there is no tomorrow — and without an ethics approval certificate. (*Editor's note.* Am I really writing this?)

   For tonight's assignment, we analyse the following question: how does the population of singles in a jazz bar fluctuate with time, if the bar is small and accommodates few patrons?

   Let $p(x,t)$ be the probability of finding x singles in the bar at time t. During the night, the bouncer allows singles to enter the bar at a constant rate K_+ (units: min^{-1}). Simultaneously, singles leave the bar at a rate K_- per single inside (units: $\mathrm{human}^{-1}\,\mathrm{min}^{-1}$), because they dislike the crowd and/or the ragged glissandi of the hungover pianist. Singles also pair up to form couples at a rate $2K_\heartsuit$ (units: $\mathrm{human}^{-2}\,\mathrm{min}^{-1}$), lulled by the smooth tones of the sax.

   (a) Put Miles on the stereo.

   (b) Argue that the master equation for $p(x,t)$ is given by

   $$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= K_+ p(x-1,t) + K_-(x+1)p(x+1,t) + K_\heartsuit(x+2)(x+1)p(x+2,t) \\ &\quad -[K_+ + K_- x + K_\heartsuit x(x-1)]p(x,t)\ . \qquad (1)\end{aligned}$$

   Explain carefully, in words, the mathematical form of each term.

   (c) By taking moments of the master equation directly, show that the mean number of singles in the bar, $\langle x\rangle$, evolves according to the deterministic rate equation

   $$\frac{d\langle x\rangle}{dt} = K_+ - K_-\langle x\rangle - 2K_\heartsuit\langle x(x-1)\rangle\ , \qquad (2)$$

   where $\langle\ldots\rangle$ denotes an ensemble average.

   (d) Show that the generating function $G(s,t) = \sum_{x=0}^{\infty} s^x p(x,t)$ satisfies the equation

   $$\frac{\partial G}{\partial t} = K_\heartsuit(1-s^2)\frac{\partial^2 G}{\partial s^2} + K_-(1-s)\frac{\partial G}{\partial s} - K_+(1-s)G\ . \qquad (3)$$

   (e) Show that the steady-state generating function is given by

   $$G_s(s) = \left(\frac{s+1}{2}\right)^{(1-2\beta)/2} \frac{I_{2\beta-1}\{[8\alpha(s+1)]^{1/2}\}}{I_{2\beta-1}[(16\alpha)^{1/2}]}\ . \qquad (4)$$

   with $\alpha = K_+/(2K_\heartsuit)$ and $\beta = K_-/(2K_\heartsuit)$. The symbol $I_\nu(\xi)$ denotes the modified Bessel function of order ν and argument ξ. [*Hint.* You may wish to change variables to $u = s+1$ and $F(u) = u^\beta G(u)$.]

   (f) From your own personal experience, or otherwise, pick sensible values of K_+, K_-, and $K_\heartsuit$. Make K_+ small enough so that the bar is sparsely filled, so that you can see the "stochastic limit" in your results below.

(g) For your chosen values, and from (4), what is the mean number of singles in the bar in the steady state?

(h) What is the standard deviation?

(i) Write a short Monte-Carlo computer program in your favourite language to simulate the evolution of $p(x,t)$. Run it several times with different random numbers. Graph a sample of your results, showing the time evolution of the population $x(t)$.

(j) How small does K_+/K_- need to be (assuming $K_\heartsuit \ll K_-$) in order to keep the fluctuations in $x(t)$ below, say, 10 per cent?

(k) Compare the steady-state output from your simulation with your analytic predictions in (g) and (h).