3 Isospin

In 1932 Heisenberg first suggested that the neutron and proton were different charge substates of the same particle, the nucleon.

$$\begin{array}{lllll} N = & p & I(N) = \frac{1}{2} & I_3 = \frac{1}{2} & \text{Proton} \\ N = & n & & I_3 = -\frac{1}{2} & \text{Neutron} \end{array}$$

He suggested this because of the closeness of the mass of the proton and neutron and because of the existance of "mirror" nuclei of nearly equal binding energy. Mirror Nuclei have equal numbers of protons and neutrons but have a neutron replaced with a proton or vice-versa. eg

^{7}Be and ^{7}Li.

This in turn led to the observation of the charge symmetry of the nuclear force.

ie The p-p and n-n forces are the same once coulomb effects have been taken into account.

Isospin is generally denoted by the symbol I, with observed component denoted by I_3.

eg. The nucleon is $I = \frac{1}{2}$, the proton is $I=\frac{1}{2}$, $I_3 = \frac{1}{2}$ and neutron $I_3 = -\frac{1}{2}$, $I = \frac{1}{2}$.

Thus the charge is given by

$$\frac{Q}{e} = \frac{1}{2} + I_3 \qquad (1)$$

for nucleons.

The whole idea of isospin is taken completely from ordinary Q.M. angular momentum. ie Spin = $\frac{1}{2}$, $J_z = \pm\frac{1}{2}$.

The formal use of isospin is that total isospin is conserved in strong interactions.

ie. The strong interaction depends only on the total isospin and is independent of it's direction, (ie. the value of I_3).

3.1 Isospin in the Pion-Nucleon System

The proton and neutron are interpreted to be two states of the isospin $\frac{1}{2}$ Nucleon.

$N(\frac{1}{2},+\frac{1}{2})$ = p, $N(\frac{1}{2},+\frac{1}{2})$=n.

The π^+, π^0, π^- are the 3 states of the isospin 1 pion. ie:

$$\pi(1,+1) = \pi^+, \pi(1,0) = \pi^0, \pi(1,-1) = \pi^-$$

We can generalize the relationship between isospin and charge by introducing the Baryon number into the formula.

$$\frac{Q}{e} = I_3 + \frac{B}{2} \tag{2}$$

Let's look at the πN system. $\pi N \rightarrow \pi N$.

$$I_\pi = 1; I_N = \frac{1}{2}; \Rightarrow \mid 1 - \frac{1}{2} \mid \leq I_{tot} \leq \mid 1 + \frac{1}{2}$$
$$\Rightarrow \frac{1}{2} \leq I_{tot} \leq \frac{3}{2}$$
$$\Rightarrow I_{tot} = \frac{1}{2} \; or \; \frac{3}{2}$$

Since the strong interaction depends on I_{tot} and not I_3 then the $2 \times 3 = 6$ reaction amplitudes can be all described terms of the I= $\frac{1}{2}$ and I=$\frac{3}{2}$ amplitudes.

Now the process:

$$\begin{array}{ccc} \pi^+ p & \rightarrow & \pi^+ p \\ (1,+1)(\frac{1}{2},+\frac{1}{2}) & & (1,+1)(\frac{1}{2},+\frac{1}{2}) \\ I_3 = \frac{3}{2} & & I_3 = \frac{3}{2} \end{array} \tag{3}$$

and

$$\begin{array}{ccc} \pi^- n & \rightarrow & \pi^- n \\ (1,-1)(\frac{1}{2},+-\frac{1}{2}) & & (1,-1)(\frac{1}{2},-\frac{1}{2}) \\ I_3 = -\frac{3}{2} & & I_3 = -\frac{3}{2} \end{array} \tag{4}$$

These reactions must be pure I= $\frac{3}{2}$ since we make I=$\frac{1}{2}$ give $I_3 = \frac{3}{2}$!)

Now let's look at the amplitudes for:

$$\begin{array}{cc} \pi^- p & \rightarrow \pi^- p \\ \pi^- p & \rightarrow \pi^0 n \\ \pi^+ n & \rightarrow \pi^+ n \\ \pi^+ n & \rightarrow \pi^0 p \end{array} \tag{5}$$

All these have $I_3 = \pm\frac{1}{2}$ and so are admixtures of $I = \frac{1}{2}$ and $I = \frac{3}{2}$ amplitudes.

The weights of the two amplitudes are given by the values of the Clebsch- Gordan coefficients.

So a state

$$\phi_1(I = \frac{1}{2}, m_1)\phi_2(I = 1, m_2) = \sum_{I=\frac{1}{2}}^{I=\frac{3}{2}} C_I\phi(I, m_1 + m_2) \tag{6}$$

We can therefore determine the isospin structure of the following composite systems.

$$\begin{aligned}
\mid \pi^+p > &= \sum_{I=\frac{1}{2}}^{I=\frac{3}{2}} C_I\phi(I, \frac{3}{2}) = 1 \mid \frac{3}{2} > && (7)\\
\mid \pi^0p > &= \sum_{I=\frac{1}{2}}^{I=\frac{3}{2}} C_I\phi(I, \frac{1}{2}) = \sqrt{\frac{2}{3}} \mid \frac{3}{2} > -\sqrt{\frac{1}{3}} \mid \frac{1}{2} > && (8)\\
\mid \pi^-p > &= \sqrt{\frac{1}{3}} \mid \frac{3}{2} > -\sqrt{\frac{2}{3}} \mid \frac{1}{2} > && (9)\\
\mid \pi^+n > &= \sqrt{\frac{1}{3}} \mid \frac{3}{2} > +\sqrt{\frac{2}{3}} \mid \frac{1}{2} > && (10)\\
\mid \pi^0n > &= \sqrt{\frac{2}{3}} \mid \frac{3}{2} > +\sqrt{\frac{2}{3}} \mid \frac{1}{2} > && (11)\\
\mid \pi^-n > &= \mid \frac{3}{2} > && (12)
\end{aligned}$$

The cross sections for the reactions are:

$$\sigma \propto \mid Amplitude \mid^2$$

Where the amplitude is the inner product between the states:

$$Amplitude =< \Psi_i \mid H \mid \Psi_f >$$

and H = interaction Hamiltonian. so

$$\sigma \propto \mid< \Psi_i \mid H \mid \Psi_f >\mid^2$$

Now the strong interaction conserves total isospin so $H\mid \frac{3}{2} > = A_3 \mid \frac{3}{2} >$ where A_3 = eigenvalue from the Hamiltonian.

$$< \frac{3}{2} \mid H \mid \frac{3}{2} >= A_3 < \frac{3}{2} \mid \frac{3}{2} >= A_3 \tag{13}$$

This is the amplitude for the $\mid \frac{3}{2} >$ interaction.
Similarly $< \frac{1}{2} \mid H \mid \frac{1}{2} >= A_1 < \frac{1}{2} \mid \frac{1}{2} >= A_1$ = Amplitude for the $\mid \frac{1}{2} >$ interaction.

Then $< \frac{3}{2} \mid H \mid \frac{1}{2} >= A_1 < \frac{3}{2} \mid \frac{1}{2} >= 0$ since there is 0 overlap between these states in the strong interaction. Of course it follows that $< \frac{1}{2} \mid H \mid \frac{3}{2} >= A_3 < \frac{1}{2} \mid \frac{3}{2} >= 0$

so

$$\sigma(\pi^+ p \to \pi^+ p) \propto \mid < \frac{3}{2} \mid H \mid \frac{3}{2} > \mid^2 = \mid A_3 \mid^2 \tag{14}$$

$$\begin{aligned}
\sigma(\pi^- p \to \pi^- p) &\propto \mid (\sqrt{\frac{1}{3}} < \frac{3}{2} \mid -\sqrt{\frac{2}{3}} < \frac{1}{2} \mid) H (\sqrt{\frac{1}{3}} \mid \frac{3}{2} > -\sqrt{\frac{2}{3}} \mid \frac{1}{2} >) \mid^2 \qquad (15)\\
&= \mid \sqrt{\frac{1}{3}} < \frac{3}{2} \mid H \mid \frac{3}{2} > -\sqrt{\frac{2}{9}} < \frac{3}{2} \mid H \mid \frac{1}{2} > \\
&- \sqrt{\frac{2}{9}} < \frac{1}{2} \mid H \mid \frac{3}{2} > + \frac{2}{3} < \frac{1}{2} \mid H \mid \frac{1}{2} > \mid^2 \\
&= \mid \frac{1}{3} A_3 + \frac{2}{3} A_1 \mid^2
\end{aligned}$$

$$\begin{aligned}
\sigma(\pi^- p \to \pi^0 n) &\propto \mid (\sqrt{\frac{1}{3}} < \frac{3}{2} \mid -\sqrt{\frac{2}{3}} < \frac{1}{2} \mid) H (\sqrt{\frac{2}{3}} \mid \frac{3}{2} > +\sqrt{\frac{1}{3}} \mid \frac{1}{2} >) \mid^2 \qquad (16)\\
&= \mid \sqrt{2}9 A_3 - \sqrt{2}9 A_1 \mid^2
\end{aligned}$$

So if $A_3 \ \geq \ A_1$ then:

$$\sigma(\pi^+ p \to \pi^+ p) : \sigma(\pi^- p \to \pi^- p) : \sigma(\pi^- p \to \pi^0 n) = 1 : \frac{1}{9} : \frac{2}{9}$$

And if $A_1 \geq \ A_3$ then:

$$\sigma(\pi^+ p \to \pi^+ p) : \sigma(\pi^- p \to \pi^- p) : \sigma(\pi^- p \to \pi^0 n) = 0 : 2 : 1$$

3.2 Strangeness and Isospin

Generalize the formula for the charge of a particle by including the strangeness Quantum Number:

$$\frac{Q}{e} = \frac{B}{2} + \frac{s}{2} + I_3 \tag{17}$$

Strangeness and Isospin are both conserved in strong interaction. The assignments of isospin to strange particles are made in the following mannor.

Baryons

s	I	I_3 = -1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	1
-1	0			Λ		
-1	1	Σ^-		Σ^0		Σ^+
-2	$\frac{1}{2}$		Ξ^-		Ξ^0	
-3	0			Ω^-		

Mesons

s	I	$-\frac{1}{2}$	$+\frac{1}{2}$
+1	$\frac{1}{2}$	k^0	k^+
-1	$\frac{1}{2}$	k^-	k^+

Strange particles were named so because they are copiously produced in strong interactions, yet they decay weakly.

generally strange particles decay in steps of $\Delta s = 1$. For example

$$\begin{array}{cccccc} \Xi^- & \rightarrow & \Lambda & + & \pi^- & \\ & & \rightarrow & p & + & \pi^- \end{array}$$

There are exceptions for example:

$$\Sigma^0 \rightarrow \Lambda + \gamma$$

Where both the initial and final states have strangeness -1. The Electromagnetic force does not conserve isospin ($I(\Sigma^0) = 1$, $I(\Lambda) = 0$), but does conserve strangeness (and all quark flavours). Different I_3's have different charges so E/M force is not conserved by rotations of I!

This is illustratrated in the different masses of different isospin projections.

Particles	Δm (MeV/c^2)	m
n - p	1.3	939
$\Sigma^0 - \Sigma^+$	3.1	1190
$\Xi^- - \Xi^0$	6.5	1318
$\Sigma^- - \Sigma^0$	4.9	1115
k^0 - $k^\pm$	4.0	495
$\pi^\pm - \pi^0$	4.6	140

The mass splittings are of the order of $\frac{|E/M|}{|strong|} \approx 10^{-2}$.

4 Invariance Principles and Conservation Laws

As established in your Quantum Mechanics lectures, an operator, Q, is conserved, (it does not change with time) if it commutes with the Hamiltonian. ie:

$$i\hbar \frac{dQ}{dt} = [Q, H] = 0 \tag{18}$$

4.1 Parity

The operation of inversion of spatial coordinates is an example of a discrete transformation.

$$P\Psi(x, y, z) = \Psi(-x, -y, -z) \quad or \quad P\Psi(\vec{r}) = \Psi(-\vec{r}) \tag{19}$$

Clearly $\mathrm{P}^2 = 1 \quad \Rightarrow \quad$ is unitary.

Now P is an interesting operator because many wavefunction are eigenstates of the P operator. If Ψ is such a wavefunction then:

$$P\Psi = \pm\Psi \tag{20}$$

If such is the case the wavefunction is said to have a well defined parity with eigen values of either +1 or -1. Some examples of wavefunctions with well defined parity's are:

$$\Psi = \cos(x), \quad P\Psi = \cos(-x) = \cos(x) = \Psi \tag{21}$$

$$\Psi = \sin(x), \quad P\Psi = \sin(-x) = -\sin(x) = -\Psi \tag{22}$$

While the wavefunction $\Psi = \cos(x) + \sin(x)$:

$$P\Psi = \cos(x) - \sin(x) \neq \pm\Psi \tag{23}$$

does not have a well defined parity.

Parity will be conserved if it commutes with the Hamiltonian. A familar example is any spherically symmetric potential. These have the property $\mathrm{H}(-\vec{r}) = \mathrm{H}(\vec{r}) = \mathrm{H}(|\,\vec{r}\,|)$

$$\Rightarrow \quad [P, H] = 0 \tag{24}$$

The bound states of these system have definate parity:

$$\begin{aligned}
[P, H]\Psi &= PH(\vec{r})\Psi(\vec{r}) - H(\vec{r})P\Psi(\vec{r}) \\
&= H(-\vec{r}\Psi(\vec{r}) - H(\vec{r})PP^{-1}P\Psi(\vec{r}) \\
&= H(\vec{r})\Psi(\vec{r}) - PH(\vec{r})P^{-1}P\Psi(\vec{r}) \\
&= H(\vec{r})\Psi(\vec{r}) - PH(\vec{r})\Psi(\vec{r}) \\
&= H(\vec{r})\Psi(\vec{r}) - H(\vec{r})\Psi(\vec{r}) \\
&= 0
\end{aligned}$$

As we found earlier all central force wavefunctions can be expressed as:

$$\Psi(r,\theta,\phi) = \chi(r)Y_l^m(\theta,\phi) \tag{25}$$

Where $Y_l^m(\theta,\phi)$ are the spherical harmics. θ is the angle to the z-axis, ϕ is the projected angle onto the x-y plane. So:

$$\Psi(r,\theta,\phi) = \chi(r)\sqrt{\frac{2(l+1)(l-m)!}{4\pi(l+m)}}p_l^m(\cos(\theta))e^{im\phi} \tag{26}$$

Where $P_l^m(\cos\theta)$ are the Legendre Polynomials.

The inversion $\vec{r} \rightarrow -\vec{r}$ is equivalent to:

$$\theta \rightarrow \pi - \theta, \quad \phi = \phi + \pi$$

$$\begin{aligned} \Rightarrow \quad P(P_l^m(\cos(\theta))e^{im\phi}) &= P_l^m(\cos(\pi-\theta))e^{im(\phi+m)} \\ &= P_l^m(-cos(\theta))e^{im\phi}e^{im\pi} \\ &= (-1)^l P_l^m(\cos\theta)e^{im\phi} \end{aligned} \tag{27}$$

So orbital A.M. l = 0,2,4 has even Parity
orbital A.M. l = 1,3,5 has odd Parity

Even l $\Rightarrow$ even Parity; odd l $\Rightarrow$ odd Parity.

Parity is a multiplicative quantum number so that the Parity of a composite system is the product of the individual Parities.

ie $\Psi = \Phi_a\Phi_b\Phi_c$.. then Parity of $\Psi = P_aP_bP_c$... then the parity of Ψ=$P_aP_bp_c$...

Strong and E/M interactions conserve parity so for example the reaction

$$p + p \rightarrow \pi^+ + p + n$$

conserves parity.

That means that we must assign an intrinsic parity to the pion.

We need to do this to ensure conservation of parity in strong interactions.

In general the parity of a system is given by:

$$P(a+b) = (-1)^l P_a \times P_b$$

ie The product of the intrinsic parties a and b $\times(-10)^l$ where l = orbital A. M.

4.2 Intrinsic Parities

Some partilces have defined intrinsic parties. For example the pion π^+, π^0, π^- has spin^{parity} 0^-,0^-,0^-. ie The pion has odd intrinsic parity.

This means the pion wavefunction changes sign under parity operations. $P \mid \pi >= - \mid \pi >$.

Therefore the parity of the two pion systems is given by the product of their parities and orbital angular momentum.

eg. $\rho^+ \rightarrow \pi^+\pi^0$ Therefore the parity of the final state is:

$$P(\pi^+)P(\pi^0)(-1)^l = (-1)(-1)(-1)^l = (-1)^l$$

Where l = orbital A.M.

The ρ^+ particle has spin 1 $\Rightarrow$ l=1 to conserve A.M. Therefore P$\mid \rho^+ >= - \mid \rho^+ >$ since strong interactions conserves parity.

4.3 The Parities of Particles and Anti-Particles

Dirac Theory predicts opposite parities for fermions and anti-fermions. This has been confirmed experimentally.

Boson anti-particles have the same parity as particles.

because fermions are always produced in pairs, the absolute parity of a fermion is a matter of convention.

The parity of a proton is defined to be even.
The parity of a neutron is defined to be even.

The strong force conserves parity.
The electromagnetic force conserves parity.
(Parity of initial state) = (Parity of the final state).

<u>The weak force does not conserve Parity</u>

One example: the K$^+$ can decay as follows:

$$K^+ \rightarrow \pi^+\pi^0$$
$$K^+ \rightarrow \pi^+\pi^-\pi^+$$

Where the angular distribution of the final state particles shows they are in an l=0 state. So the K^+ can decay to a state of parity (-1) (-1) = (+1) and to a state of (-1) (-1) (-1) = (-1) ie. both +ve and -ve parities. Clearly this means that whatever parity the k^+ might have, it is not preserved following weak decays.

There are many other examples in the weak interaction of parity violation. A beautiful example is main decay of the muon:

$$\mu^+ \rightarrow e^+ \bar{\nu_\mu} \nu_e$$

Where the positron is preferentially emitted along the direction of the muon spin. This is only possible if parity is violated.

A consequence of parity violation in the weak interaction is that neutrinos do not have a well defined parity.

Neutrinos are always left handed (their spin points opposite their direction of motion). Anti-neutrinos are always right handed. (Spins points in the direction of motion.)

This is beautifully demonstrated in π-decay.

$$\pi^+ \rightarrow \mu^+ + \nu_\mu$$

The spin of the μ^+ is always opposite its direction of motion. (Can tell by the direction of the e^+ from $\mu^+ \rightarrow e^+ \bar{\nu_\mu} \nu_e$).

We find that the spin of the μ^- is in its direction of motion.

4.4 Charge Conjugation Invariance

Charge Conjugation transformations (represented by the operator C) means changing a particle to it's anti-particle.

ie. C| proton > = | anti-proton >, C| p > = | $\bar{p}$ >
C| electron > = | positron >, C| e^- > = | e^+ >
C| pion > = | anti-pion >, C| π^- > = | π^+ >

Anti-particles have precisely the same mass and exactly opposite charge and magnetic moment to particles. Positive magnetic moment means the magnetic moment is aligned with the spin direction. Negative magnetic moement means the spin direction is opposite the direction of motion.

4.5 Eignestates of the C-operator

Consider the operation of C on the charged pion wavefunction.

$$C \mid \pi >= \eta \mid \pi^- > \neq \eta' \mid \pi^+ >$$

So charged wavefunctions are not eigenstates of the C-operator.

In fact only systems of particles and anti-particles or self-conjugate particles can be eigenstates of C. For example:

$$\mid \gamma >, \mid \pi^0 >, \mid \pi^+\pi^- >$$

Now $C \mid \gamma >= - \mid \gamma >$ because the γ mediates the E/M fields change sign by the C-operator. In the case of the $\mid \pi^0 >$ we note that:

$$\pi^0 \rightarrow \gamma\gamma$$

and because C is a multiplicative quantum number, the final state has positive C-parity.

$$C \mid \gamma >\mid \gamma >= C \mid \gamma > C \mid \gamma >= + \mid \gamma >\mid \gamma >$$

so C$\mid \pi^0 >=\mid \pi^0 >$

In general we write:

$$J^{PC} = 0^{-+}$$

Where 0 = particles spin, - refers to the particles' parity and + is the particles C-parity.

C is conserved in strong and Electromagnetic but is not conserved by the weak interaction.

Example Neutrinos:

Both the P and C operations on neutrinos induce states that are not observed in nature. However the combined operation CP does produce observed states.

Further examples:

$$\begin{array}{rcll} \Gamma(\pi^+ \rightarrow \mu^+\nu_{\mu l}) & \neq & \Gamma(\pi^+ \rightarrow \mu^+\nu_{\mu r}) = 0 & Parity\ Violation \\ \Gamma(\pi^+ \rightarrow \mu^+\nu_{\mu l}) & \neq & \Gamma(\pi^- \rightarrow \mu^+\bar{\nu_{\mu l}}) = 0 & C\ Violation \\ \Gamma(\pi^+ \rightarrow \mu^+\nu_{\mu l}) & \neq & \Gamma(\pi^- \rightarrow \mu^+\bar{\nu_{\mu r}}) = 0 & CP\ Invariance \end{array}$$

So the combined operations CP are conserved in weak interactions (except for a small 10^{-3} violation in kaon decays (and maybe some B decays too).

4.6 Time Reversal invariance

So for example, any collision looks identical with t forward or backward.

Experimental tests of T invariance have been made in strong interaction processes.

$$p +^{27} Al \rightarrow \alpha +^{24} Mg, \quad \alpha +^{24} Mg \rightarrow p +^{27} Al$$

Which are observed to be identical after the spin statistic differences between the initial and final states and taken into account.

4.7 CPT invariance

The CPT theorem states that all interactions are invariant under C, P and T transformations taken in any order.

The proof of this theorem rests on very general assumptions about Quantum Field Theory (In QFT antiparticles are particles backwards in time)

It is very difficult to construct a Quantum Field Theory that is not CPT invariant.

The best tests of CPT come from properties of particles and anti-particles.

lifetime	Fractional difference
$\tau_{\pi^+} - \tau_{\pi^-}$	$< 10^{-3}$
$\tau_{\mu^+} - \tau_{\mu^-}$	$< 10^{-4}$
Magnetic Moment	Fractional difference
$\mu^+ - \mu^-$	$< 10^{-8}$
$e^+ - e^-$	$< 10^{-10}$
Mass	Fractional difference
$m_{\bar{p}}$ - m_p	$< 10^{-4}$
m_{K^0} - $m_{\bar{K^0}}$	$< 10^{-18}$

This last very sensitive test is possible because of the unique properties of the neutral kaon system